A ~~Complete~~ Partial Solution Manual for the Problems in

Matthew D. Schwartz's "*Quantum Field Theory and the Standard Model*" [1]

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Part I

Field theory ✓

Chapter 1

Microscopic Theory of Radiation

This chapter does not include any problems.

Chapter 2

Lorentz invariance and second quantization

2.1

- **0th order:**

  For $v = 0$, the transformations are simply

$$x \to x_0, \tag{2.1}$$
$$t \to t_0. \tag{2.2}$$

  It is trivially true that $t^2 - x^2 \equiv t_0^2 - x_0^2$ is preserved.

- **1st order:**

  At $\mathcal{O}(v^1)$ order, we start from $x \to x_0 + \delta x_1$ and $t \to t_0 + \delta t_1$, where we already know that $\delta x_1 = vt_0$. By assumption, δt_1 is of order $\mathcal{O}(v^1)$. Since the transformation must preserve $t^2 - x^2 \equiv t_0^2 - x_0^2$, we obtain

$$\begin{aligned} t^2 - x^2 \to t_0^2 + 2t_0\delta t_1 - x_0^2 - 2vx_0t_0 &\equiv t_0^2 - x_0^2 \\ \implies \delta t_1 = \frac{2vx_0t_0}{2t_0} &= vx_0, \end{aligned} \tag{2.3}$$

  where terms of order $\mathcal{O}(v^2)$ or higher are neglected.

- **2nd order:**

  Proceeding to $\mathcal{O}(v^2)$, we set

$$x \to x_0 + vt_0 + \delta x_2, \tag{2.4}$$
$$t \to t_0 + vx_0 + \delta t_2, \tag{2.5}$$

  where δx_2 and δt_2 are of order $\mathcal{O}(v^2)$. Substituting into the invariant quantity,

$$\begin{aligned} t^2 - x^2 \to t_0^2 + v^2x_0^2 + 2vx_0t_0 + 2t_0\delta t_2 - x_0^2 - v^2t_0^2 - 2vx_0t_0 - 2x_0\delta x_2 &\equiv t_0^2 - x_0^2 \\ \implies 2(x_0\delta x_2 - t_0\delta t_2) = v^2(x_0^2 - t_0^2), \end{aligned} \tag{2.6}$$

where terms of order $\mathcal{O}(v^3)$ or higher are neglected. As we assume δx_2 and δt_2 are both linear in x_0 and t_0, and of order $\mathcal{O}(v^2)$, the only consistent solution is

$$\delta x_2 = \frac{1}{2}v^2 x_0, \tag{2.7}$$
$$\delta t_2 = \frac{1}{2}v^2 t_0. \tag{2.8}$$

- The expansion of $\frac{1}{\sqrt{1-v^2}} = (1-v^2)^{-\frac{1}{2}}$ follows as

$$1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots,$$

leading to the approximations valid for $v \ll 1$

$$x \to \frac{x+vt}{\sqrt{1-v^2}} = x + vt + \frac{1}{2}v^2 x + \ldots \tag{2.9}$$
$$t \to \frac{t+vx}{\sqrt{1-v^2}} = t + vx + \frac{1}{2}v^2 t + \ldots. \tag{2.10}$$

These results are consistent with the perturbation expansions derived above, matching order by order in powers of v.

2.2

(a) At the CM frame (which, for the LHC, also coincides with the lab frame), each of the two colliding protons has an energy of $E_p = 7$ TeV. Given that $m_p \approx 0.938$ GeV $\ll E_p$, we find

$$\gamma m_p = E_p$$
$$\gamma = \frac{E_p}{m_p}$$
$$\frac{1}{1-v^2} = \frac{E_p^2}{m_p^2}$$
$$v = \sqrt{1 - \frac{m_p^2}{E_p^2}} \approx 1 - \frac{m_p^2}{2E_p^2}. \tag{2.11}$$

Clearly, the quantity $\frac{m_p^2}{2E_p^2}$ represents the deviation of the proton's speed from the speed of light:

$$\frac{m_p^2}{2E_p^2} \approx 8.98 \times 10^{-9}\, c \approx 2.69\ \mathrm{m\,s^{-1}} = \boxed{9.68\ \mathrm{km\,h^{-1}}}, \tag{2.12}$$

where I include a factor of the speed of light $c = 299792458\ \mathrm{m\,s^{-1}}$ to restore the correct dimension of velocity.

(b) Consider the rest frame of proton 1. The lab frame moves with velocity $v_{lab} = v$ relative to proton 1's rest frame, where v is given by Eq. (2.11). Proton 2, meanwhile, moves with speed $v_{2,lab} = v$ relative to the lab frame.

Applying the collinear velocity addition formula,

$$v_{rel} = \frac{v_{lab} + v_{2,lab}}{1 + v_{lab}v_{2,lab}} = \frac{2v}{1+v^2} \approx \frac{2 - \frac{m_p^2}{E_p^2}}{1 + 1 - \frac{m_p^2}{E_p^2}} + \mathcal{O}\left(\frac{m_p^4}{E_p^4}\right) \approx 1 = c. \tag{2.13}$$

Thus, one proton is moving at nearly the speed of light c relative to the other.

2.3

(a) From Eq. (1.6) of the textbook, we know that the total energy of CMB photons in the universe is given by

$$E_{\text{CMB, tot}} = \frac{V}{\pi^2} \int_0^\infty \frac{E^3}{e^{\beta E} - 1} dE. \tag{2.14}$$

On the other hand, the total energy is also related to the number density by

$$E_{\text{CMB, tot}} \equiv \int_0^\infty n(E) E dE, \tag{2.15}$$

where $n(E)$ is the number density of photons as a function of energy. The total number of CMB photons in the universe is given by

$$N_{\text{CMB, tot}} \equiv \int_0^\infty n(E) dE. \tag{2.16}$$

Comparing Eq. (2.14) with Eq. (2.15), we conclude

$$N_{\text{CMB, tot}} \equiv \int_0^\infty n(E) dE = \frac{V}{\pi^2} \int_0^\infty \frac{E^2}{e^{\beta E} - 1} dE. \tag{2.17}$$

We can then calculate the average energy $\langle E_{\text{CMB}} \rangle$ of CMB photons as

$$\begin{aligned} \langle E_{\text{CMB}} \rangle = \frac{E_{\text{CMB, tot}}}{N_{\text{CMB, tot}}} &\equiv \frac{\int_0^\infty \frac{E^3}{e^{\beta E}-1} dE}{\int_0^\infty \frac{E^2}{e^{\beta E}-1} dE} \\ &= \frac{1}{\beta} \frac{\int_0^\infty \frac{\varepsilon^3}{e^{\varepsilon}-1} d\varepsilon}{\int_0^\infty \frac{\varepsilon^2}{e^{\varepsilon}-1} d\varepsilon} \\ &= \frac{1}{\beta} \frac{\Gamma(4)\zeta(4)}{\Gamma(3)\zeta(3)} \\ &\approx 3 k_B T_{\text{CMB}} \times \frac{\pi^4}{90} \times \frac{1}{1.202} \\ &= \boxed{6.35 \times 10^{-4} \text{ eV}}, \end{aligned} \tag{2.18}$$

where I redefine the variable $\varepsilon \equiv \beta E$ in the second line and use the property of the Riemann zeta function,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Also, I used $\zeta(4) = \frac{\pi^4}{90}$, and $\zeta(3) \approx 1.202$. In the following, I shall denote $\langle E_{\mathrm{CMB}} \rangle$ simply as E_{CMB}.

(b) To find the threshold energy for pion production, we consider the lab frame as where the initial proton and the initial photon are collinear and head-on. We set

$$p_{p,i} = (E_p, \vec{p}_p), \tag{2.19}$$
$$p_\gamma = (E_{\mathrm{CMB}}, \vec{p}_\gamma), \tag{2.20}$$

where $|\vec{p}_\gamma| = E_{\mathrm{CMB}}$. In the CM frame, the final pion and proton are at rest:

$$p_{p,f} = (m_p, 0), \tag{2.21}$$
$$p_\pi = (m_\pi, 0). \tag{2.22}$$

Note that the four-momentum in the initial and final states above are not measured in the same frame. However, the squared sum of four-momenta, i.e., the invariant mass squared, is a Lorentz scalar, so we can evaluate it in either frame. Thus,

$$\begin{aligned} (p_{p,i} + p_\gamma)^2 &= (p_{p,f} + p_\pi)^2 \\ m_p^2 + 2\left(E_{CMB}E_p - |\vec{p}_p||\vec{p}_\gamma| \cos\theta\right) &= m_p^2 + m_\pi^2 + 2m_p m_\pi, \\ m_p^2 + 2E_{CMB}(E_p + |\vec{p}_p|) &= m_p^2 + m_\pi^2 + 2m_p m_\pi, \end{aligned} \tag{2.23}$$

where I have used the four-momentum conservation in the first line, the initial proton and CMB photon collide head-on ($\cos\theta = \cos\pi = -1$) and that $|\vec{p}_\gamma| = E_{\mathrm{CMB}}$. Given that $m_p \approx 0.938$ GeV and $|\vec{p}_p| = \sqrt{E_p^2 - m_p^2}$, the threshold energy is then derived by

$$\begin{aligned} E_p + \sqrt{E_p^2 - m_p^2} &= \frac{m_\pi^2 + 2m_p m_\pi}{2E_{CMB}} \\ E_p^2 - m_p^2 &= E_p^2 + \left(\frac{m_\pi^2 + 2m_p m_\pi}{2E_{CMB}}\right)^2 - 2E_p\left(\frac{m_\pi^2 + 2m_p m_\pi}{2E_{CMB}}\right) \\ E_p &= \frac{m_\pi^2 + 2m_p m_\pi}{4E_{CMB}} + \frac{m_p^2 E_{CMB}}{m_\pi^2 + 2m_p m_\pi} \\ &\approx \boxed{1.07 \times 10^{11}\ \mathrm{GeV}}\ . \end{aligned} \tag{2.24}$$

(c) The proton-pion system has an invariant mass of $M_{\mathrm{inv}} = m_p + m_\pi$. In the lab frame, the total energy of this system is $E_{\mathrm{tot}} = E_p + E_{\mathrm{CMB}}$ due to energy conservation. To find the relative velocity between the CM frame and the lab frame, we calculate the Lorentz factor γ that relates the two frames as

$$\gamma = \frac{E_{\mathrm{tot}}}{M_{\mathrm{inv}}} = \frac{E_p + E_{\mathrm{CMB}}}{m_p + m_\pi}. \tag{2.25}$$

Since the outgoing proton is at rest in the CM frame, this is also the Lorentz factor of the outgoing proton in the lab frame. Thus, in the lab frame, its energy is

$$E_{p,f} = \gamma m_p = \frac{E_p + E_{CMB}}{m_p + m_\pi} m_p \approx \boxed{9.35 \times 10^{10}\ \text{GeV}}\ . \tag{2.26}$$

2.4

Yes, the transformation Y is indeed a Lorentz transformation. It can be achieved by first performing a spatial rotation of an angle $\theta = \pi$ around the y-axis, followed by the parity transformation P:

$$(t, x, y, z) \xrightarrow{R_y(\pi)} (t, -x, y, -z) \xrightarrow{P} (t, x, -y, z). \tag{2.27}$$

Since the group product ($Y = P \circ R_y(\pi)$) is closed, Y is also an element of Lorentz group, i.e., a Lorentz transformation. Because it can be generated by other group generators, we do not treat it as an independent discrete Lorentz transformation alongside P and T.

2.5

(a) The energy range of X-rays is approximately $\mathcal{O}(0.1) - \mathcal{O}(100)$ keV, which is much greater than the typical ionization energy of an electron in most crystals (i.e., $\mathcal{O}(1) - \mathcal{O}(10)$ eV). Therefore, the electron can essentially be treated as free.

(b) The initial electron is at rest, and we apply four-momentum conservation:

$$\begin{aligned} p_e + p_\gamma &= p'_e + p'_\gamma \\ p'^2_e &= (p_e + p_\gamma - p'_\gamma)^2 \\ m_e^2 &= m_e^2 + 2p_e \cdot p_\gamma - 2p_e \cdot p'_\gamma - 2p_\gamma \cdot p'_\gamma \\ 0 &= m_e E_\gamma - m_e E'_\gamma - E_\gamma E'_\gamma (1 - \cos\theta) \\ E'_\gamma &= \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e}(1 - \cos\theta)} \\ \Aboxed{f'_\gamma &= \frac{f_\gamma}{1 + \frac{h f_\gamma}{m_e c^2}(1 - \cos\theta)}}\ , \end{aligned} \tag{2.28}$$

where primes denote final-state variables, and θ is the scattering angle between the initial and final photons. I also insert Planck's constant h and the speed of light c in the final form to restore the correct dimensions. The plot is shown in Fig. 2.1.

(c) From Eq. (2.28), as $m_e \to 0$, $f'_\gamma \to 0$, except when $\theta = 0$. At this angle, the photon's frequency—and consequently, its energy—remains unchanged, and no interaction occurs between the photon and the electron. *Note that a massless particle cannot be at rest in any reference frame, meaning the previous derivation is not valid in this limit (technically, one should really think the limit* $\boldsymbol{E_\gamma / m_e \gg 1}$*).*

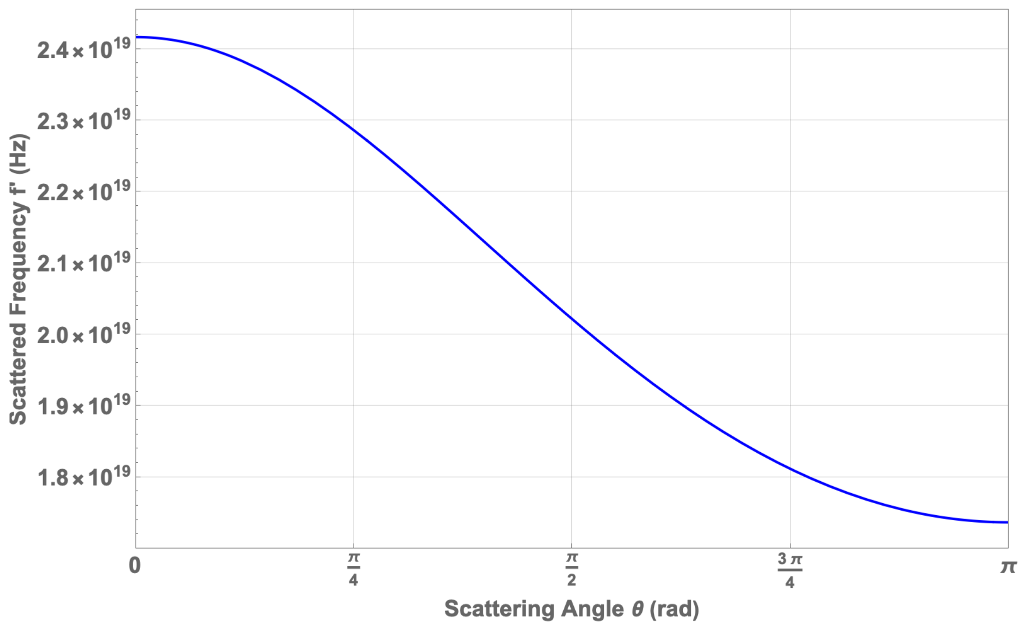


Fig. 2.1: The y-axis represents the scattering frequency f', while the x-axis represents the scattering angle θ. The initial photon energy is fixed as $E_\gamma = 100$ keV

(d) Classically (i.e., if the photon momenta is not quantized), the frequency of the outgoing radiation is identical to that of the incoming radiation, and the distribution is simply a constant line. This can be seen by taking the $h \to 0$ limit in Eq. (2.28). The classical physical picture, which you might recall from your E&M class, is the incoming radiation induces the electron to oscillate at the same frequency as the radiation. The oscillation of the electron then releases away radiation at the same frequency. Hence, $f_\gamma = f'_\gamma$.

2.6

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\Theta(k^0) &= \int_{-\infty}^{\infty} dk^0 \delta((k^0)^2 - \omega_k^2)\Theta(k^0) \\
&= \int_{-\infty}^{\infty} dk^0 \frac{\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)}{2\omega_k}\Theta(k^0) \\
&= \int_{0}^{\infty} dk^0 \frac{\delta(k^0 - \omega_k)}{2\omega_k} \\
&= \boxed{\frac{1}{2\omega_k}},
\end{aligned} \tag{2.29}$$

where I used the identity $\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$, in which i runs over all roots of the argument of the delta function $g(x)$.

(b) Recall from Eq. (2.11) of the textbook, the Lorentz transformation Λ is defined by

$$\Lambda^T g \Lambda = g. \tag{2.30}$$

Taking the determinant of both sides,

$$\begin{aligned} \det\left(\Lambda^T \eta \Lambda\right) &= \det \eta \\ (\det \Lambda)^2 \det \eta &= \det \eta \\ |\det \Lambda| &= 1, \end{aligned} \tag{2.31}$$

where I used $\det \eta = -1$.

A coordinate transformation induces a change in the integration measure according to

$$dk^0 dk^1 \cdots dk^n = |\det \mathcal{J}| dk^{0'} dk^{1'} \cdots dk^{n'}, \tag{2.32}$$

where $\mathcal{J}$ is the Jacobian of the transformation. Since a Lorentz transformation is itself a coordinate transformation, the Lorentz transformation matrix Λ is nothing but the Jacobian $\mathcal{J}$. Hence,

$$\boxed{d^4k = |\det \Lambda| d^4k' = d^4k'} \,. \tag{2.33}$$

Therefore, the measure d^4k is trivially Lorentz invariant.

Side Remark: In a curved spacetime[a], the metric g varies from point to point, and no global Lorentz transformation Λ can satisfy $\Lambda^T g \Lambda = g'$ universally. Nevertheless, the Einstein equivalence principle guarantees the existence of a locally flat reference frame at every point, where the laws of physics reduce to those of special relativity (i.e. $g' \to \eta$). Hence, a generalization of Eq. (2.31) becomes

$$\begin{aligned} \det\left(\Lambda^T g \Lambda\right) &= \det \eta \\ (\det \Lambda)^2 \det g &= -1 \\ |\det \Lambda| &= \sqrt{-\det g}, , \end{aligned} \tag{2.34}$$

and the generalization of the Lorentz transformation of the differential measure becomes

$$d^4k = \sqrt{-\det g}\; d^4k'. \tag{2.35}$$

This is where the $\sqrt{-\det g}$ factor in the Einstein-Hilbert Lagrangian comes from (see Eqs. (8.145), (8.146), and (22.26) of the textbook).

[a] This is why I intentionally used the Minkowski metric notation η in Eq. (2.31) instead of the general metric notation g as in the textbook.

(c)

$$\begin{aligned} \int \frac{d^3k}{2\omega_k} &= \int d^3k \int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2)\Theta(k^0) \\ &= \int d^4k \delta(k^2 - m^2)\Theta(k^0) \end{aligned} \tag{2.36}$$

by part (a). Now,

- The integral measure d^4k is Lorentz invariant by part (b).
- The function $\delta(k^2 - m^2)$ is Lorentz invariant since its argument is a scalar.
- Also, the on-shell condition by the delta function defines a two-sheeted hyperboloid in k-space, either $k_0 > 0$ or $k_0 < 0$. However, for a proper orthochronous Lorentz transformation, it's impossible to transform a point from one sheet to the other. Hence, the sign of k^0 is unambiguous and $\Theta(k^0)$ is Lorentz invariant[1].

As both the integral measure and the integrand are Lorentz invariant, the entire expression is Lorentz invariant.

2.7

(a)

$$\partial_z(e^{-za^\dagger} a e^{za^\dagger}) = -a^\dagger e^{-za^\dagger} a e^{za^\dagger} + e^{-za^\dagger} a a^\dagger e^{za^\dagger} = e^{-za^\dagger}(aa^\dagger - a^\dagger a)e^{za^\dagger} = e^{-za^\dagger}\left[a, a^\dagger\right]e^{za^\dagger} = 1. \tag{2.37}$$

(b) Notice that when $z = 0$, $e^{-za^\dagger} a e^{za^\dagger} = a$. Using this boundary condition to integrate Eq. (2.37), one can obtain

$$e^{-za^\dagger} a e^{za^\dagger} = z + a. \tag{2.38}$$

Then,

$$a\left|z\right\rangle = a e^{za^\dagger}\left|0\right\rangle = e^{za^\dagger} e^{-za^\dagger} a e^{za^\dagger}\left|0\right\rangle = e^{za^\dagger}(z+a)\left|0\right\rangle = z e^{za^\dagger}\left|0\right\rangle = z\left|z\right\rangle. \tag{2.39}$$

Thus, $|z\rangle$ is an eigenstate of a with eigenvalue $\boxed{z}$.

(c)

$$\begin{aligned} \langle n|\hat{N}|z\rangle &= \langle n|a^\dagger a|z\rangle \\ n\langle n|z\rangle &= z\sqrt{n}\langle n-1|z\rangle \\ \langle n|z\rangle &= \frac{z}{\sqrt{n}}\langle n-1|z\rangle \\ c_n &= \frac{z}{\sqrt{n}}c_{n-1}. \end{aligned} \tag{2.40}$$

Also, the base case is

$$c_0 \equiv \langle 0|z\rangle = \langle 0|e^{za^\dagger}|0\rangle = \langle 0|\left(1 + za^\dagger + \frac{1}{2!}(za^\dagger)^2 + \cdots\right)|0\rangle = 1. \tag{2.41}$$

Then, by induction,

$$\boxed{\langle n|z\rangle \equiv c_n = \frac{z^n}{\sqrt{n!}}}. \tag{2.42}$$

[1] Recall that when we say something is Lorentz invariant, we really mean it's invariant under a proper orthochronous Lorentz transformation.

(d) We will need the following relations:

$$\langle z|z\rangle = \sum_n \langle z|n\rangle \langle n|z\rangle = \sum_n |c_n|^2 = \sum_n \frac{|z|^{2n}}{n!} = e^{|z|^2}, \tag{2.43}$$

$$\langle z| a |z\rangle = z \langle z|z\rangle = z e^{|z|^2}, \tag{2.44}$$

$$\langle z| a^\dagger a |z\rangle = |z|^2 \langle z|z\rangle = |z|^2 e^{|z|^2}, \tag{2.45}$$

$$\langle z| a a^\dagger |z\rangle = \langle z| (1 + a^\dagger a) |z\rangle = (1 + |z|^2) e^{|z^2|}, \tag{2.46}$$

$$\langle z| a^2 |z\rangle = z^2 e^{|z|^2}. \tag{2.47}$$

Then,

$$\langle x\rangle = \frac{\langle z| x |z\rangle}{\langle z|z\rangle} = \frac{1}{\sqrt{2m\omega}} \frac{\langle z| (a + a^\dagger) |z\rangle}{\langle z|z\rangle} = \frac{1}{\sqrt{2m\omega}}(z + z^*), \tag{2.48}$$

$$\langle x^2\rangle = \frac{1}{2m\omega} \frac{\langle z| (aa + a^\dagger a^\dagger + a a^\dagger + a^\dagger a) |z\rangle}{\langle z|z\rangle} = \frac{1}{2m\omega}(z^2 + z^{*2} + 2|z|^2 + 1), \tag{2.49}$$

$$\langle p\rangle = \frac{\langle z| p |z\rangle}{\langle z|z\rangle} = i\sqrt{\frac{m\omega}{2}} \frac{\langle z| (a^\dagger - a) |z\rangle}{\langle z|z\rangle} = i\sqrt{\frac{m\omega}{2}}(z^* - z), \tag{2.50}$$

$$\langle p^2\rangle = -\frac{m\omega}{2} \frac{\langle z| (aa + a^\dagger a^\dagger - a a^\dagger - a^\dagger a) |z\rangle}{\langle z|z\rangle} = -\frac{m\omega}{2}(z^2 + z^{*2} - 2|z|^2 - 1). \tag{2.51}$$

Thus,

$$\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2m\omega}, \tag{2.52}$$

$$\Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\omega}{2}. \tag{2.53}$$

and therefore,

$$\boxed{\Delta p \Delta x = \sqrt{\Delta x^2 \Delta p^2} = \frac{1}{2}}. \tag{2.54}$$

(e) Suppose there exists such an eigenstate $|\beta\rangle \equiv \sum_n b_n |n\rangle$ of $a^\dagger$, with a nontrivial eigenvalue $\beta \neq 0$. Then,

$$a^\dagger |\beta\rangle = \beta |\beta\rangle = \sum_n \beta b_n |n\rangle . \tag{2.55}$$

We also have

$$a^\dagger |\beta\rangle = \sum_n b_n \sqrt{n+1} |n+1\rangle = \sum_n b_{n-1} \sqrt{n} |n\rangle , \tag{2.56}$$

where I have shifted the index in the last step.

Taking the difference of the two expressions Eq. (2.55) and Eq. (2.56), we obtain the recursion relation

$$\begin{aligned} \beta b_n - \sqrt{n} b_{n-1} &= 0 \\ b_n = b_{n-1} \frac{\sqrt{n}}{\beta} &= b_0 \frac{\sqrt{n!}}{\beta^n}, \end{aligned} \tag{2.57}$$

where the last step follows from induction. However, note that

$$0 = \langle 0| a^\dagger |\beta\rangle = \beta b_0 \implies b_0 = 0 \tag{2.58}$$

Thus, all coefficients $b_n = 0$, indicating an eigenstate of $a^\dagger$ cannot exist.

Chapter 3

Classical field theory

3.1

$$
\begin{aligned}
0 = \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta(\partial_\nu \partial_\mu \phi) + \dots \right] \\
&= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)} \right) + \cdots \right] \delta\phi \\
&\quad + \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta(\partial_\nu \phi) \right) - \partial_\nu \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta\phi \right) + \cdots \right] \\
&= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi)} \right) + \cdots \right] \delta\phi \\
&\quad + \int d^4x \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) + 2 \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta(\partial_\nu \phi) \right) - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta\phi \right) + \cdots \right].
\end{aligned}
\tag{3.1}
$$

The second line is a total derivative and vanishes if one assumes that the fields and their derivatives vanish at spatial and temporal infinity. The first first line must vanish for arbitrary variations $\delta\phi$, which leads to the generalized Euler–Lagrange equation for a Lagrangian of the form $\mathcal{L}[\phi, \partial_\mu \phi, \partial_\nu \partial_\mu \phi, \cdots]$

$$
\boxed{\sum_{i=0}^{n} \sum_{\mu_1 \leq \cdots \leq \mu_i} (-1)^i \partial_{\mu_1} \cdots \partial_{\mu_i} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu_1 \cdots \mu_i}} \right) = 0}\,, \tag{3.2}
$$

where I have adopted the notation commonly used in General Relativity context: $\phi_{,\mu_1 \cdots \mu_i} \equiv \partial_{\mu_1} \cdots \partial_{\mu_i} \phi$.

3.2

(a) Lorentz symmetry implies that replacing the scalar field $\phi(x^\mu)$ with $\phi((\Lambda^{-1})^\mu{}_\nu x^\nu)$—its form under a Lorentz-transformed frame—leaves physical observables (like the Lagrangian, the equations of motion, etc) invariant. For a proper infinitesimal Lorentz transformation, we can write

$$
\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \omega^\mu_{\ \nu}, \tag{3.3}
$$

where $\omega_{\mu\nu}$ is antisymmetric (see Eq. (10.13) of the textbook for example). To prove this, we can start from the defining condition for Lorentz transformations (cf. Eq. (2.41)): $\Lambda^T g \Lambda = g$. Since the transformation must reduce to the identity for infinitesimal ω, we can expand

$$\begin{aligned}
\Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} &= g_{\alpha\beta} \\
(\delta^{\mu}_{\alpha} + \omega^{\mu}{}_{\alpha}) g_{\mu\nu} (\delta^{\nu}_{\beta} + \omega^{\nu}{}_{\beta}) &= g_{\alpha\beta} \\
g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} &= g_{\alpha\beta} \\
\omega_{\alpha\beta} &= -\omega_{\beta\alpha},
\end{aligned} \tag{3.4}$$

where higher orders of ω is dropped since we assume it to be infinitesimal. Hence, the infinitesimal expansion is proven.

Expanding the field to linear order in $\omega_{\mu\nu}$, we have

$$\begin{aligned}
\phi(x^{\alpha}) &\to \phi((\Lambda^{-1})_{\rho}{}^{\alpha} x^{\rho})) \\
&= \phi(x^{\alpha} - \omega_{\rho}{}^{\alpha} x^{\rho}) \\
&= \phi(x) - \omega_{\rho}{}^{\alpha} x^{\rho} \partial_{\alpha} \phi(x) \\
&= \phi(x) - g_{\beta\rho} \omega^{\beta\alpha} x^{\rho} \partial_{\alpha} \phi(x) \\
&= \phi(x) - \frac{1}{2}(g_{\beta\rho} \omega^{\beta\alpha} x^{\rho} \partial_{\alpha} - g_{\beta\rho} \omega^{\alpha\beta} x^{\rho} \partial_{\alpha}) \phi(x) \\
&= \phi(x) - \frac{1}{2}(g_{\beta\rho} \omega^{\beta\alpha} x^{\rho} \partial_{\alpha} - g_{\beta\alpha} \omega^{\rho\beta} x^{\alpha} \partial_{\rho}) \phi(x),
\end{aligned} \tag{3.5}$$

where I relabeled $\rho \leftrightarrow \alpha$ in the last term in the last line. Then,

$$\frac{\delta \phi}{\delta \omega^{\mu\nu}} = -\frac{1}{2}(g_{\beta\rho} g_{\beta\mu} g_{\alpha\nu} x_{\rho} \partial_{\alpha} - g_{\beta\alpha} g_{\rho\mu} g_{\beta\nu} x_{\alpha} \partial_{\rho}) \phi(x) = \frac{1}{2}(x_{\nu} \partial_{\mu} - x_{\mu} \partial_{\nu}) \phi(x). \tag{3.6}$$

Since the Lagrangian $\mathcal{L}$ itself is also a scalar, it transforms similarly:

$$\frac{\delta \mathcal{L}}{\delta \omega^{\mu\nu}} = \frac{1}{2}(x_{\nu} \partial_{\mu} \mathcal{L} - x_{\mu} \partial_{\nu} \mathcal{L}) = \frac{1}{2}[\partial_{\alpha}(x_{\nu} g_{\alpha\mu} - x_{\mu} g_{\alpha\nu})] \mathcal{L}. \tag{3.7}$$

On the other hand, by invoking the equations of motion, the variation of the Lagrangian can also be written as

$$\frac{\delta \mathcal{L}[\phi_n, \partial_{\alpha} \phi_n]}{\delta \omega^{\mu\nu}} = \partial_{\alpha} \left(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} \phi_n)} \frac{\delta \phi_n}{\delta \omega^{\mu\nu}} \right) = \partial_{\alpha} \left(\frac{1}{2} \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} \phi_n)} (x_{\nu} \partial_{\mu} - x_{\mu} \partial_{\nu}) \phi_n \right). \tag{3.8}$$

Equating Eq. (3.7) and Eq. (3.8), we find

$$\begin{aligned}
\partial_{\alpha} \left[x_{\nu} \left(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} \phi_n)} \partial_{\mu} \phi_n - g_{\alpha\mu} \mathcal{L} \right) - x_{\mu} \left(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} \phi_n)} \partial_{\nu} \phi_n - g_{\alpha\nu} \mathcal{L} \right) \right] &= 0 \\
\partial_{\alpha}[x_{\nu} \mathcal{T}_{\alpha\mu} - x_{\mu} \mathcal{T}_{\alpha\nu}] &= 0,
\end{aligned} \tag{3.9}$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor. Thus, we can identify the conserved currents as

$$\boxed{K_{\mu\nu\alpha} = x_{\nu} \mathcal{T}_{\alpha\mu} - x_{\mu} \mathcal{T}_{\alpha\nu}}\ . \tag{3.10}$$

(b) Given the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi, \tag{3.11}$$

we calculate the energy-momentum tensor as

$$\mathcal{T}_{\alpha\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}\partial_\mu \phi - g_{\alpha\mu}\mathcal{L} = (\partial_\alpha\phi)(\partial_\mu\phi) - \frac{1}{2}g_{\alpha\mu}[(\partial_\nu\phi)^2 - m^2\phi^2] \tag{3.12}$$

Note first that the energy-momentum tensor itself is conserved when the equation of motion of the field is satisfied:

$$\begin{aligned} \partial_\alpha \mathcal{T}_{\alpha\mu} &= \Box\phi(\partial_\mu\phi) + (\partial_\alpha\phi)(\partial_\alpha\partial_\mu\phi) - (\partial_\mu\partial_\nu\phi)(\partial_\nu\phi) + m^2(\partial_\mu\phi)\phi \\ &= \Box\phi(\partial_\mu\phi) + m^2(\partial_\mu\phi)\phi \\ &= 0, \end{aligned} \tag{3.13}$$

where I invoked the equation of motion of the field $\Box\phi = -m^2\phi$.

Next, we verify the conservation of the Lorentz current $K_{\mu\nu\alpha}$:

$$\boxed{\partial_\alpha K_{\mu\nu\alpha} = \partial_\alpha[x_\nu \mathcal{T}_{\alpha\mu} - x_\mu \mathcal{T}_{\alpha\nu}] = g_{\nu\alpha}\mathcal{T}_{\alpha\mu} - g_{\mu\alpha}\mathcal{T}_{\alpha\nu} = \mathcal{T}_{\nu\mu} - \mathcal{T}_{\mu\nu} = 0}\ , \tag{3.14}$$

where I have used Eq. (3.13), the identity $\partial_\mu x_\nu = g_{\mu\nu}$, and also the fact that the energy-momentum tensor is symmetric (cf. Eq. (3.12)).

(c)

$$Q_i = \int d^3x K_{0i0} = \int d^3x(x_i\mathcal{T}_{00} - t\mathcal{T}_{0i}) = \int d^3x(x_i\mathcal{E} - tp_i) = \int d^3x x_i \mathcal{E} - tP_i. \tag{3.15}$$

Here, $\mathcal{E}$ is the energy density, p_i is the momentum density, and P_i is the total momentum of the system. These conserved quantities correspond exactly to the three boost generators of the Lorentz group[1]. Comparing this result directly with Eq. (10.22) of the textbook, one can observe that these quantities Q_i are precisely the boost generators L_{0i} of the Lorentz group in momentum space.

(d) From the Heisenberg equation for a conserved charge operator,

$$0 = \frac{dQ_i}{dt} = i[Q_i, H] + \frac{\partial Q_i}{\partial t}, \tag{3.16}$$

one readily observes this can be consistent if and only if

$$\boxed{i\frac{\partial Q_i}{\partial t} = [Q_i, H]}\ . \tag{3.17}$$

If the charge operator is not invariant under the equations of motion (i.e., $[Q_i, H] \neq 0$), the Heisenberg equation still holds. However, in this case, the charge operator has explicit

[1] From here onward, I will slightly abuse terminology by using the word "charges" to refer both to the symmetry group generators themselves and to their eigenvalues (the conserved charges arising from Noether's theorem).

time dependence—it has intrinsic dynamics, which is exactly why it is not an invariant of the equations of motion. Therefore, although these charges remain conserved (due to Noether's theorem), they do not correspond to invariants of the dynamics.

As a concrete example, for the boost generators $Q_i \equiv L_{0i}$ we just calculated, explicit time dependence appears from Eq. (3.15). Applying the Heisenberg equation Eq. (3.17), one finds:

$$\boxed{[L_{0i}, H] = -iP_i} \ . \tag{3.18}$$

This is precisely one of the Poincaré algebra relation, where the commutator of a boost generator in i-th direction and the time-translation operator (the Hamiltonian H) yields the i-th spatial momentum operator P_i. Physically, this commutation relation indicates that under a boost transformation, the energy of the system changes (i.e., frame-dependent) unless the system possesses zero spatial momentum—in that special case, the system's energy is exactly its invariant mass.

On the other hand, for rotations, the Poincaré algebra states that the rotation generators J_i commute with the time-translation operator (the Hamiltonian H):

$$\boxed{[J_i, H] = 0} \ . \tag{3.19}$$

Therefore, one can define a conserved charge associated with rotations—the spin—which also remains invariant under the equations of motion. Thus, rotation provides a well-defined quantum number labeling representations of the full spacetime symmetry – Poincaré group, while boost can not.

3.3

(a) Let

$$\mathcal{L}' \equiv \mathcal{L} + \partial_\alpha X_\alpha. \tag{3.20}$$

Under a spacetime translation, following the textbook treatment, $\phi_n \to \phi_n + \xi^\nu \partial_\nu \phi_n$, we compute the variation of the action:

$$\delta S = \int d^4x \delta\mathcal{L}' = \int d^4x [\delta\mathcal{L} + \delta(\partial_\alpha X_\alpha)] = \int d^4x [\delta\mathcal{L} + \partial_\alpha \delta X_\alpha]. \tag{3.21}$$

Since $X_\alpha[\phi_n, \partial_\mu\phi_n]$ is a functional of the fields ϕ_n and fields' derivatives $\partial_\mu\phi_n$[2], its variation is

$$\delta X_\alpha = \sum_n \left[\frac{\partial X_\alpha}{\partial\phi_n}\delta\phi_n + \frac{\partial X_\alpha}{\partial(\partial_\beta\phi_n)}\partial_\beta\delta\phi_n\right], \tag{3.22}$$

and thus, its functional derivative with respect to the transformation parameter ξ^ν becomes

$$\begin{aligned}\frac{\delta X_\alpha}{\delta\xi^\nu} &= \sum_n \left[\frac{\partial X_\alpha}{\partial\phi_n}\partial_\nu\phi_n + \frac{\partial X_\alpha}{\partial(\partial_\beta\phi_n)}\partial_\beta(\partial_\nu\phi_n)\right] \\ &= \sum_n \left[\frac{\partial X_\alpha}{\partial\phi_n}\partial_\nu\phi_n + \frac{\partial X_\alpha}{\partial(\partial_\beta\phi_n)}\partial_\nu(\partial_\beta\phi_n)\right] \\ &= \partial_\nu X_\alpha,\end{aligned} \tag{3.23}$$

which simply follows from the chain rule.

Invoking the equations of motion, we now have

$$\begin{aligned}\frac{\delta\mathcal{L}'[\phi_n,\partial_\mu\phi_n]}{\delta\xi^\nu} &= \frac{\delta\mathcal{L}[\phi_n,\partial_\mu\phi_n]}{\delta\xi^\nu} + \partial_\alpha\frac{\delta X_\alpha[\phi_n,\partial_\mu\phi_n]}{\delta\xi^\nu} \\ &= \partial_\mu\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n\right) + \partial_\alpha\partial_\nu X_\alpha \\ &= \partial_\mu\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n + \partial_\nu X_\mu\right),\end{aligned} \tag{3.24}$$

where I have relabeled the dummy indices $\alpha \to \mu$ in the second term.

Meanwhile, since $\mathcal{L}'$ is also a scalar, we expect

$$\frac{\delta\mathcal{L}'}{\delta\xi^\nu} = \partial_\nu(\mathcal{L} + \partial_\alpha X_\alpha). \tag{3.25}$$

Equating Eq. (3.24) and Eq. (3.25), we find

$$\partial_\mu\left(\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial_\nu\phi_n + \partial_\nu X_\mu - g_{\mu\nu}\mathcal{L} - g_{\mu\nu}\partial_\alpha X_\alpha\right) = 0. \tag{3.26}$$

[2]

Side Remark: As a spoiler, the famous θ term $\varepsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} = \partial^\mu(\varepsilon_{\mu\nu\alpha\beta}(A^a_\nu F^a_{\alpha\beta} - \frac{g}{3}f^{abc}A^a_\nu A^b_\alpha A^c_\beta))$ is an example of total derivative on the Lagrangian, which does not contribute to matrix element in *perturbation theory*, but has real physics effect in *non-perturbative theory*. See, for example, Eq. (7.109), Eq. (29.105), and Eq. (30.89) of the textbook. Also note the functional – **the Chern-Simons current** – $X^\mu = \varepsilon_{\mu\nu\alpha\beta}(A^a_\nu F^a_{\alpha\beta} - \frac{g}{3}f^{abc}A^a_\nu A^b_\alpha A^c_\beta)$ of the θ term indeed depends both on ***fields*** as well as ***fields derivatives***. Also, this chapter is about *classical fields*, which have no fundamental reasons to argue X_μ should not depend on *fields derivatives*. I have seen many false derivations of this problem assuming X_μ depends only on *fields*, which is not valid in the very first place. In fact, as I showed with chain rule, the result holds irregardless of how higher-derivatives of the fields $\partial_\alpha\partial_\beta\cdots\phi_n$ the functional X_μ depends on.

Therefore, the change in the energy-momentum tensor due to adding a total derivative to the Lagrangian is given by

$$\boxed{\delta\mathcal{T}_{\mu\nu} = \partial_\nu X_\mu - g_{\mu\nu}\partial_\alpha X_\alpha = \partial_\alpha\left(g_{\alpha\nu}X_\mu - g_{\mu\nu}X_\alpha\right) \equiv \partial_\alpha K_{\alpha\mu\nu}}, \tag{3.27}$$

where I have written it as a total derivative of an auxiliary tensor $K_{\alpha\mu\nu}$. Note that $K_{\alpha\mu\nu}$ is antisymmetric in its first two indices: $\alpha \leftrightarrow \mu$.

(b) Evaluating the resulting variation of the total energy,

$$\begin{aligned} \delta Q &= \int d^3x\delta\mathcal{T}_{00} \\ &= \int d^3x\left(\partial_t X_0 - \partial_\alpha X_\alpha\right) \\ &= \int d^3x\left(\partial_t X_0 - \partial_t X_0 + \partial_i X_i\right) \\ &= \int d^3x\left(\partial_i X_i\right) \\ &= \int_{d\Omega} X_i dA \\ &= 0, \end{aligned} \tag{3.28}$$

by divergence theorem and assuming X_i are constructed from operators dying fast enough at spatial infinity.

(c) We first compute

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} - \frac{\partial(\partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)} = g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu}. \tag{3.29}$$

For the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2$, the field equation for A_ν becomes

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}\partial_\mu[F_{\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})] = -\partial_\mu F_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial A_\nu} = 0, \tag{3.30}$$

where we used the fact that $F_{\mu\nu} = -F_{\nu\mu}$.

The canonical energy-momentum tensor is then

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\gamma)}\partial_\nu A_\gamma - g_{\mu\nu}\mathcal{L} \\ &= -\frac{1}{2}F_{\alpha\beta}(g_{\alpha\mu}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\mu})(\partial_\nu A_\gamma) - g_{\mu\nu}\mathcal{L} \\ &= -\frac{1}{2}(F_{\mu\gamma} - F_{\gamma\mu})(\partial_\nu A_\gamma) - g_{\mu\nu}\mathcal{L} \\ &= -F_{\mu\alpha}(\partial_\nu A_\alpha) - g_{\mu\nu}\mathcal{L}. \end{aligned} \tag{3.31}$$

Clearly, the first term of the tensor is not symmetric under $\mu \leftrightarrow \nu$, so we aim to correct this.

To symmetrize, we consider the antisymmetric part and require it to vanish after adding the contribution from $\delta T_{\mu\nu}$ given by Eq. (3.27):

$$\begin{aligned}0 &= T_{\mu\nu} + \delta T_{\mu\nu} - T_{\nu\mu} - \delta T_{\nu\mu} \\ &= -F_{\mu\alpha}(\partial_\nu A_\alpha) + F_{\nu\alpha}(\partial_\mu A_\alpha) + \partial_\alpha(K_{\alpha\mu\nu} - K_{\alpha\nu\mu}) \\ &= (\partial_\alpha A_\mu - \partial_\mu A_\alpha)(\partial_\nu A_\alpha) + (\partial_\nu A_\alpha - \partial_\alpha A_\nu)(\partial_\mu A_\alpha) + \partial_\alpha(K_{\alpha\mu\nu} - K_{\alpha\nu\mu}) \\ &= (\partial_\alpha A_\mu)(\partial_\nu A_\alpha) - (\partial_\alpha A_\nu)(\partial_\mu A_\alpha) + \partial_\alpha(K_{\alpha\mu\nu} - K_{\alpha\nu\mu}) \\ &= (\partial_\alpha A_\mu)(\partial_\nu A_\alpha) - (\partial_\alpha A_\nu)(\partial_\mu A_\alpha) + (\partial_\alpha A_\nu)(\partial_\alpha A_\mu) - (\partial_\alpha A_\nu)(\partial_\alpha A_\mu) + \partial_\alpha(K_{\alpha\mu\nu} - K_{\alpha\nu\mu}) \\ &= F_{\nu\alpha}(\partial_\alpha A_\mu) + F_{\alpha\mu}(\partial_\alpha A_\nu) + \partial_\alpha(K_{\alpha\mu\nu} - K_{\alpha\nu\mu}) \\ &= \partial_\alpha(F_{\nu\alpha}A_\mu - F_{\mu\alpha}A_\nu + K_{\alpha\mu\nu} - K_{\alpha\nu\mu}),\end{aligned} \tag{3.32}$$

where we have used the equation of motion of the fields $\partial_\mu F_{\mu\nu} = 0$ from Eq. (3.30) to get the last line.

Given the antisymmetric property of the first two indices of $K_{\alpha\mu\nu}$ and also the antisymmetric property of $F_{\mu\alpha}$, a natural choice is

$$K_{\alpha\mu\nu} = F_{\mu\alpha}A_\nu. \tag{3.33}$$

The above derivation then ensures cancellation of the antisymmetric parts in the canonical energy-momentum tensor. After including the divergence of the auxiliary tensor $K_{\alpha\mu\nu}$, the modified energy-momentum tensor becomes

$$\boxed{T_{\mu\nu} = -F_{\mu\alpha}(\partial_\nu A_\alpha) - g_{\mu\nu}\mathcal{L} + \partial_\alpha K_{\alpha\mu\nu} = -F_{\mu\alpha}(\partial_\nu A_\alpha) - g_{\mu\nu}\mathcal{L} + F_{\mu\alpha}(\partial_\alpha A_\nu) = F_{\mu\alpha}F_{\alpha\nu} - g_{\mu\nu}\mathcal{L}}, \tag{3.34}$$

which is now manifestly symmetric.

Finally, to solve for X_μ, we start from where we introduced the auxiliary tensor and plugging the explicit form of Eq. (3.33) we just found,

$$\begin{aligned}\partial_\alpha\left(g_{\alpha\nu}X_\mu - g_{\mu\nu}X_\alpha\right) \equiv \partial_\alpha K_{\alpha\mu\nu} &= \partial_\alpha(F_{\mu\alpha}A_\nu) \\ \partial_\nu X_\mu - g_{\mu\nu}\partial_\alpha X_\alpha &= F_{\mu\alpha}\partial_\alpha A_\nu.\end{aligned} \tag{3.35}$$

Now, contracting both sides with $g_{\mu\nu}$:

$$\begin{aligned}\partial_\alpha X_\alpha - 4\partial_\alpha X_\alpha &= g_{\mu\nu}F_{\mu\alpha}\partial_\alpha A_\nu \\ \partial_\alpha X_\alpha &= \frac{1}{3}F_{\alpha\mu}\partial_\alpha A_\mu = \frac{1}{3}\partial_\alpha(F_{\alpha\mu}A_\mu) \\ X_\alpha &= \frac{1}{3}F_{\alpha\mu}A_\mu.\end{aligned} \tag{3.36}$$

A word of caution however, is that it is important to note that this expression for X_α is obtained through contraction with the metric tensor $g_{\mu\nu}$, and a general inversion to get such a term in Lagrangian while preserving Lorentz covariance might not exist. Note that even A^μ itself is not uniquely determined due to gauge invariance. In fact, what people usually do is to add the remedy term directly onto the energy-momentum tensor instead of the Lagrangian. This is totally fine as long as the modification does not spoils the energy-momentum conservation. It's the conservation law what really physical.

3.4

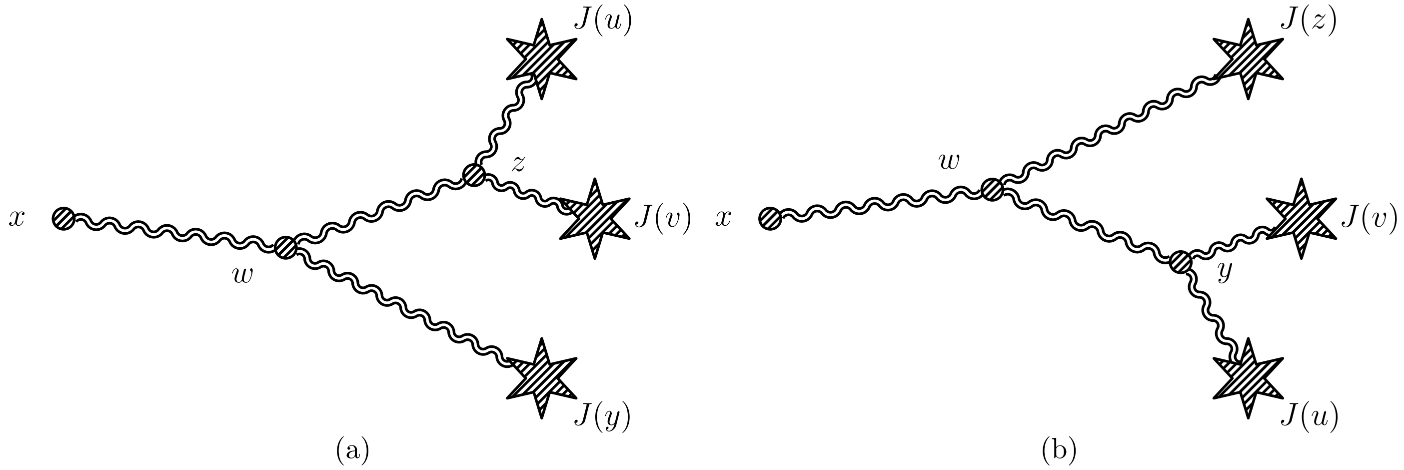


Fig. 3.1: The two next-order Feynman diagrams of classical gravity.

The two next-order diagrams are displayed in Fig. 3.1. Reading off the expressions directly from the diagrams, we have:

$$\begin{aligned} h_2(x) = & -\lambda^2 \int d^4w \int d^4z \int d^4u \int d^4v \int d^4y \; \Pi(x,w)\Pi(w,y)\Pi(w,z)\Pi(z,u)\Pi(z,v)J(u)J(v)J(y) \\ & -\lambda^2 \int d^4w \int d^4z \int d^4u \int d^4v \int d^4y \; \Pi(x,w)\Pi(w,z)\Pi(w,y)\Pi(y,u)\Pi(y,v)J(u)J(v)J(z) \\ = & \boxed{-2\lambda^2 \int d^4w \int d^4z \int d^4u \int d^4v \int d^4y \; \Pi(x,w)\Pi(w,y)\Pi(w,z)\Pi(z,u)\Pi(z,v)J(u)J(v)J(y)}, \end{aligned} \tag{3.37}$$

where in the last step we used the fact that y and z are dummy integration variables.

From the Green's function method, let us write $h = h_0 + h_1 + h_2$, where h_2 is of order $\mathcal{O}(\lambda^2)$. Then, the equation of motion is

$$\Box(h_0 + h_1 + h_2) - \lambda(h_0 + h_1 + h_2)^2 - J = 0, \tag{3.38}$$

which implies at order $\mathcal{O}(\lambda^2)$,

$$\Box h_2 = 2\lambda h_0 h_1 + \mathcal{O}(\lambda^3), \tag{3.39}$$

Thus, we have

$$h_2 = 2\lambda \frac{1}{\Box}(h_0 h_1) = 2\lambda \frac{1}{\Box}\left[\left(\frac{1}{\Box}J\right)\left(\lambda \frac{1}{\Box}\left(\frac{1}{\Box}J \frac{1}{\Box}J\right)\right)\right]. \tag{3.40}$$

Using the two-point Green's function $\Pi = -\frac{1}{\Box}$, this becomes

$$\boxed{h_2(x) = -2\lambda^2 \int d^4w \int d^4z \int d^4u \int d^4v \int d^4y \; \Pi(x,w)\Pi(w,y)\Pi(w,z)\Pi(z,u)\Pi(z,v)J(u)J(v)J(y)}, \tag{3.41}$$

which confirms the result obtained directly from the Feynman diagrams.

3.5

(a) The equation of motion is

$$\Box\phi - m^2\phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{3.42}$$

The constant solutions $\phi(x) = c$ can be found by plugging this back into the equation of motion,

$$-m^2 c + \frac{\lambda}{3!}c^3 = 0 \tag{3.43}$$

$$c\left(-m + \sqrt{\frac{\lambda}{6}}c\right)\left(m + \sqrt{\frac{\lambda}{6}}c\right) = 0. \tag{3.44}$$

Thus, the constant solutions are $\boxed{c = 0}$ and $\boxed{c = \pm\sqrt{\frac{6m^2}{\lambda}}}$. For these constant configurations, the kinetic term vanishes, so the ground state energy can be determined by evaluating the potential energy $V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$:

$$\boxed{V(\phi = 0) = 0,\ V\left(\phi = \pm\sqrt{\frac{6m^2}{\lambda}}\right) = -\frac{3}{2}\frac{m^4}{\lambda}}. \tag{3.45}$$

The two solutions $\phi = \pm\sqrt{\frac{6m^2}{\lambda}}$ correspond to degenerate ground states.

(b) The $\mathbb{Z}_2$ symmetry transformation $\phi \to -\phi$ maps one ground state to the other. If the field has a vacuum expectation value $\langle\phi\rangle = c$, this transformation changes it as $\langle\phi\rangle \to -c$. Therefore, the vacuum does not respect the symmetry, unless $c = 0$.

(c) Consider the field redefinition $\phi(x) = c + \pi(x)$. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}(c + \pi(x))\Box\pi(x) + \frac{1}{2}m^2(c + \pi(x))^2 - \frac{\lambda}{4!}(c + \pi(x))^4 \tag{3.46}$$

The equation of motion for $\pi(x)$ is

$$\Box\pi(x) - m^2(c + \pi(x)) + \frac{\lambda}{6}(c + \pi(x))^3 = 0. \tag{3.47}$$

Observe that the terms without $\pi(x)$ are: $-m^2c + \frac{\lambda}{6}c^3 = c\left(-m^2 + \frac{6\lambda m^2}{6\lambda}\right) = 0$ cancel out, where we plugged in $c^2 = \frac{6m^2}{\lambda}$ for either of the degenerate vacua. Thus, $\pi(x) = 0$ solves the equation of motion.

Under the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$, we have

$$\begin{aligned} c + \pi &\to -c - \pi, \\ \pi &\to -\pi - 2c \end{aligned} \tag{3.48}$$

Since this transformation is really just $\phi \to -\phi$ written in another way, the Lagrangian for $\pi(x)$ of course remains invariant. Indeed, explicitly substituting $\pi \to -\pi - 2c$ into Eq. (3.46) gives

$$\mathcal{L} \to -\frac{1}{2}(-c - \pi(x))\Box(-\pi(x)) + \frac{1}{2}m^2(-c - \pi(x))^2 - \frac{\lambda}{4!}(-c - \pi(x))^4 = \mathcal{L}. \tag{3.49}$$

3.6

(a) We have calculated the equation of motion for a massless A_ν in Eq (3.30). Adding the mass term and the current term, the equation of motion for A_ν becomes

$$\begin{aligned} \partial_\mu F_{\mu\nu} + m^2 A_\nu &= J_\nu \\ \Box A_\nu - \partial_\nu \partial_\mu A_\mu + m^2 A_\nu &= J_\nu. \end{aligned} \tag{3.50}$$

Taking the derivative ∂_ν on both sides gives

$$m^2(\partial_\nu A_\nu) = \partial_\nu J_\nu = 0. \tag{3.51}$$

Thus, when $m \neq 0$, this enforces the Lorenz gauge condition $\boxed{\partial_\mu A_\mu = 0}$.

(b) The derivation follows exactly as in Eq. (3.61) and Eq. (3.62) of the textbook, with the substitution $\Box \to \Box + m^2$. Therefore, in Fourier space, we directly find

$$\begin{aligned} A_0(r) &= \int \frac{d^3k}{(2\pi)^3} \frac{e}{k^2+m^2} e^{i\vec{k}\cdot\vec{r}} \\ &= \frac{e}{(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \frac{1}{k^2+m^2} e^{ikr\cos\theta} \\ &= \frac{e}{4\pi^2} \int_0^\infty dk \frac{k^2}{k^2+m^2} \frac{e^{ikr} - e^{-ikr}}{ikr} \\ &= \frac{e}{4\pi^2} \left(\int_0^\infty dk \frac{k^2}{k^2+m^2} \frac{e^{ikr}}{ikr} + \int_{-\infty}^{0} dk \frac{k^2}{k^2+m^2} \frac{e^{ikr}}{ikr} \right) \\ &= \boxed{\frac{e}{4\pi^2 ir} \int_{-\infty}^{\infty} \frac{kdk}{k^2+m^2} e^{ikr}}, \end{aligned} \tag{3.52}$$

where, in the next-to-last line, we redefined $k \to -k$ in the second term.

(c) The integrand has poles at $k = \pm im$. For e^{ikr}, we need to close the contour in the upper half-plane, which captures the pole at $k = im$ as $m > 0$. Evaluating the residue gives

$$\boxed{A_0(r) = \frac{e}{4\pi^2 ir}(2\pi i)\frac{im}{im+im}e^{-mr} = \frac{e}{4\pi r}e^{-mr}}. \tag{3.53}$$

(d) Evidently, in the limit $m \to 0$, this expression recovers the familiar Coulomb potential:

$$\boxed{A_0(r) = \frac{e}{4\pi r}}. \tag{3.54}$$

(e) The Yukawa potential above exhibits a characteristic range given by $r \sim \frac{1}{m}$. Given that the typical range of the nuclear force is approximately 1 fm, this suggests that

$$\boxed{m \sim 1\ \text{fm}^{-1} = 1\ \text{fm}^{-1} \times (\hbar c) \approx 200\ \text{MeV}}. \tag{3.55}$$

(f) Imposing the condition $\partial_\mu A_\mu = 0$ directly in the Lagrangian, we find

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu\end{aligned} \tag{3.56}$$

The equation of motion for A_ν then becomes

$$\begin{aligned}(\Box + m^2)A_\nu &= J_\nu \\ (\Box + m^2)(\partial_\nu A_\nu) &= \partial_\nu J_\nu \\ 0 &= 0.\end{aligned} \tag{3.57}$$

Hence, current conservation is automatically satisfied. Originally, it is the mass term that effectively serves as a Lagrange multiplier enforcing this constraint. If the mass term is switched off, the constraint in Eq. (3.51) reduces to a trivial equality.

3.7

(a) Since the action $\mathcal{S} = \int d^4x\, \mathcal{L}$ must be dimensionless in natural units, it follows that $[\mathcal{L}] = 4$. Examining the kinetic term in the Lagrangian, $-\frac{1}{2}h\Box h$, we find:

$$\begin{aligned}2[h] + [\Box] = [\mathcal{L}] &= 4 \\ 2[h] + 2 &= 4 \\ \boxed{[h] = 1}\ .\end{aligned} \tag{3.58}$$

Considering the second term, $(M_{\text{Pl}})^a h^2 \Box h$, this implies:

$$\begin{aligned}a[M_{\text{Pl}}] + 3[h] + [\Box] = [\mathcal{L}] &= 4 \\ a + 3 + 2 &= 4 \\ \boxed{a = -1}\ .\end{aligned} \tag{3.59}$$

Moreover, from Eq. (3.35) of the textbook, we know $[T] = [\mathcal{L}] = 4$. For the third term, $-(M_{\text{Pl}})^b hT$, we have:

$$\begin{aligned}b[M_{\text{Pl}}] + [h] + [T] &= 4 \\ b + 1 + 4 &= 4 \\ \boxed{b = -1}\ .\end{aligned} \tag{3.60}$$

(b) At first order in the source, $h^{(1)} \sim \mathcal{O}(T^1)$, the equation of motion reads:

$$\begin{aligned}\Box h^{(1)} &= -\frac{T}{M_{\text{Pl}}} = -\frac{m}{M_{\text{Pl}}}\delta^3(x), \\ h^{(1)}(x) &= -\frac{m}{M_{\text{Pl}}}\frac{1}{\Box}\delta^3(x).\end{aligned} \tag{3.61}$$

This has the same structure as the Coulomb potential (cf. Eq. (3.61) in the textbook), except for a different constant prefactor. Upon Fourier transformation, we obtain:

$$h^{(1)} = -\frac{m}{M_{\text{Pl}}}\frac{1}{r}. \tag{3.62}$$

The factor of 4π is absorbed into a rescaling of the gravitational field $h(x)$ to align with the classical Newtonian potential.

Proceeding to second order in the source, $h^{(2)} \sim \mathcal{O}(T^2)$:

$$\Box h^{(2)} = \frac{1}{M_{\text{Pl}}}\Box\left[(h^{(1)})^2\right],$$
$$\boxed{h^{(2)} = \frac{m^2}{M_{\text{Pl}}^3}\frac{1}{r^2}}. \tag{3.63}$$

(c) Since the classical gravitational force is given by the gradient of the potential, for circular motion, the orbital frequency ω relates to the gravitational potential via (noting that an extra factor of $\frac{1}{M_{\text{Pl}}}$ on the right hand side is needed to ensure dimensional consistency, since $[\omega] = [\text{s}^{-1}] = 1$):

$$\omega^2 R = \frac{1}{M_{\text{Pl}}}\left[\frac{dh^{(1)}}{dr}\right]_{r=R}$$
$$\omega^2 = \frac{1}{M_{\text{Pl}}R}\left[\frac{dh^{(1)}}{dr}\right]_{r=R} = \frac{m_{\text{Sun}}}{M_{\text{Pl}}^2 R^3} = \frac{G_N m_{\text{Sun}}}{R^3} \tag{3.64}$$
$$\boxed{\omega \approx 0.8 \times 10^{-7}\ \text{s}^{-1}}.$$

(d) The correction to the orbital frequency is:

$$\begin{aligned}\delta\omega &= \frac{1}{2\omega}|\delta(\omega^2)| \\ &= \frac{1}{2\omega}\frac{1}{M_{\text{Pl}}R}\left|\frac{dh^{(2)}}{dr}\right|_{r=R} \\ &= \frac{1}{\omega}\frac{m_{\text{Sun}}^2}{M_{\text{Pl}}^4 R^4} \\ &= \frac{1}{\omega}\frac{G_N^2 m_{\text{Sun}}^2}{R^4 c^2} \\ &\approx 2.1 \times 10^{-14}\ \text{s}^{-1} \\ &= \boxed{86\ \text{arcsec/century}},\end{aligned} \tag{3.65}$$

where we have inserted a factor of $1/c^2$ to restore dimensions.

(e) Accounting for the influence of other planets:

$$\begin{aligned}\delta\omega &= \frac{1}{2\omega_{\text{solar}}}\sum_{\text{planets}}\frac{1}{M_{\text{Pl}}R_{\text{planets}}}\left|\frac{dh^{(1)}}{dr}\right|_{r=R_{\text{planets}}}\\ &= \frac{G_N}{2\omega_{\text{solar}}}\sum_{\text{planets}}\frac{m_{\text{planets}}}{R^3_{\text{planets}}}\\ &\approx 4\times 10^{-12}\ \text{s}^{-1}\\ &= \boxed{2\times 10^4\ \text{arcsec/century}}\,,\end{aligned} \tag{3.66}$$

where we approximate the distances between Mercury and the other planets by the planetary orbital radii relative to the Sun.

(f) Both corrections are indeed observable in the case of Mercury, and constitute key experimental tests of general relativity. In fact, at the time of writing, even the analogous corrections for Venus are detectable. See Ref. [2] for details.

(g) Using the general Euler-Lagrange equations Eq. (3.2) derived in Problem 3.1, the equation of motion for $h(x)$ reads:

$$\Box h = (M_{\text{Pl}})^{-1}[2h\Box h + \Box(h^2) - T]. \tag{3.67}$$

The additional term is $(M_{\text{Pl}})^{-1}2h\Box h$. Since this term is of the same order as $(M_{\text{Pl}})^{-1}\Box(h^2)$, and given our rough order-of-magnitude estimation, both contribute comparably and can be safely neglected.

3.8

The blackbody radiation paradox essentially argues that if the electromagnetic field is treated classically, the number of available modes can increase without bound as the frequency grows. Consequently, the total energy density, when integrated over all modes, becomes infinite. This divergence can only be avoided if the higher-frequency modes are exponentially suppressed, as shown in Eq. (2.14) and Eq. (2.17). Therefore, to resolve this issue, the electromagnetic field must be treated quantum mechanically.

This argument remains valid even when electrons and atoms are described quantum mechanically, i.e., with discrete energy levels. Suppose we only quantized the electrons and atoms, while electromagnetic field remains classical. First note that classically, each mode of the electromagnetic field carries an energy of $\sim kT$ by the equipartition theorem, regardless of the frequency. This suggests that, even though the electrons or atoms system is quantized, it is still capable of emitting radiation at any frequency across all modes. This again leads to a divergence when integrating over an infinite number of modes.

In principle, the same reasoning applies to gravity. There is a clear analogy between the electromagnetic and gravitational wave, especially as gravitational wave is now an experimentally verified observation by LIGO and Virgo [3]. However, due to the extremely weak coupling of gravitational interactions, whether a similar blackbody radiation equivalence can actually be observed is another matter.

3.9

(a) The Lagrangian is[3]

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - J_\mu A_\mu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)^2 - J_\mu A_\mu \\ &= \frac{1}{2}A_\mu \Box A_\mu - J_\mu A_\mu,\end{aligned} \tag{3.68}$$

where we have imposed the Lorenz gauge condition $\partial_\mu A_\mu = 0$. Referring to Section 3.4 of the textbook, under the Lorentz gauge, the equation of motion for A_μ reduces to

$$\Box A_\mu = J_\mu, \tag{3.69}$$

or

$$A_\mu = \frac{J_\mu}{\Box}. \tag{3.70}$$

Substituting this back into the Lagrangian Eq. (3.68), and applying the Fourier space replacement $\Box \to -k^2$ (cf. Eq. (3.60) of the textbook), we find

$$\boxed{\mathcal{L} = J'_\mu \frac{1}{2k^2} J_\mu} \qquad \text{(momentum space)}. \tag{3.71}$$

(b) In momentum space, the current conservation condition $\partial_\mu J_\mu = 0$ becomes

$$\begin{aligned} ik_\mu \cdot J_\mu &= 0 \\ i(k_0 J_0 - k_1 J_1 - k_2 J_2 - k_3 J_3) &= 0. \end{aligned} \tag{3.72}$$

Choosing $k_\mu = (\omega, \kappa, 0, 0)$, this implies

$$\boxed{J_1 = \frac{\omega}{\kappa} J_0}\,. \tag{3.73}$$

(c) Substituting this relation back into the interaction term, we have

$$\begin{aligned} J'_\mu \frac{1}{2k^2} J_\mu &= \frac{J'_0 J_0 - J'_1 J_1 - J'_2 J_2 - J'_3 J_3}{2(\omega^2 - \kappa^2)} \\ &= \boxed{-\frac{J'_0 J_0}{2\kappa^2} - \frac{J'_2 J_2 + J'_3 J_3}{2(\omega^2 - \kappa^2)}}\,. \end{aligned} \tag{3.74}$$

(d) The first term, which lacks time derivatives (since no ω dependence) i.e., no dynamics, corresponds to Eq. (3.61) of the textbook, describing a stationary point charge at the origin—namely, the classical Coulomb potential. This term is instantaneous and, as such, non-causal. The remaining terms, involving time derivatives, describe the two causally propagating physical degrees of freedom. The poles at $\omega = \pm\kappa$ in these terms correspond to the advanced and retarded solutions of classical electrodynamics.

[3] It's likely that the Lagrangian given in Problem 3.9 of the textbook contains a wrong sign in the current term. This appears inconsistent with, for example, Eq. (3.87) or Eq. (8.98) of the textbook, although the physics remains unaffected by this sign.

(e) The instantaneous term represents an unphysical degree of freedom, which can be eliminated by an appropriate choice of gauge. Following the method outlined in Section 8.2 of the textbook, one can adopt the Coulomb gauge to set $J_0 = 0$. Consequently, all physical observables remain causal, and and no communication faster than the speed of light can be made.

3.10

(a) The Lagrangian is[4]

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} + \frac{1}{M_{\text{Pl}}} h_{\mu\nu} T_{\mu\nu}. \tag{3.75}$$

The equation of motion for $h_{\mu\nu}$ is

$$\Box h_{\mu\nu} = -\frac{1}{M_{\text{Pl}}} T_{\mu\nu}, \tag{3.76}$$

or

$$h_{\mu\nu} = -\frac{1}{M_{\text{Pl}}} \frac{T_{\mu\nu}}{\Box}. \tag{3.77}$$

Replacing this back to the Lagrangian Eq. (3.75) and applying the Fourier space replacement $\Box \to -k^2$ (cf. Eq. (3.60) of the textbook), we find

$$\boxed{\mathcal{L} = T'_{\mu\nu} \frac{1}{2k^2} T_{\mu\nu}} \qquad \text{(momentum space).} \tag{3.78}$$

(b) Since $T_{\mu\nu}$ is, by assumption, a symmetric rank-2 tensor, it contains 10 independent components in general. Writing this explicitly, we have

$$\boxed{\begin{aligned} \mathcal{L} = {} & T'_{00} \frac{1}{2k^2} T_{00} - T'_{01} \frac{1}{k^2} T_{01} - T'_{02} \frac{1}{k^2} T_{02} - T'_{03} \frac{1}{k^2} T_{03} \\ & + T'_{11} \frac{1}{2k^2} T_{11} + T'_{12} \frac{1}{k^2} T_{12} + T'_{13} \frac{1}{k^2} T_{13} + T'_{22} \frac{1}{2k^2} T_{22} + T'_{23} \frac{1}{k^2} T_{23} + T'_{33} \frac{1}{2k^2} T_{33}. \end{aligned}} \tag{3.79}$$

(c) In momentum space, the current conservation condition $\partial_\mu T_{\mu\nu} = 0$ becomes

$$\begin{gathered} ik_\mu \cdot T_{\mu\nu} = 0 \\ i(k_0 T_{0\nu} - k_1 T_{1\nu} - k_2 T_{2\nu} - k_3 T_{3\nu}) = 0. \end{gathered} \tag{3.80}$$

Choosing $k_\mu = (\omega, \kappa, 0, 0)$, we deduce

$$T_{1\nu} = \frac{\omega}{\kappa} T_{0\nu}, \tag{3.81}$$

which gives four constraints for $\nu = 0, 1, 2, 3$. Using the symmetric property of $T_{\mu\nu}$, we similarly have

$$T_{\mu 1} = \frac{\omega}{\kappa} T_{\mu 0} \tag{3.82}$$

[4] It's likely that the Lagrangian given in Problem 3.10 of the textbook contains a wrong sign in the kinetic term. This appears inconsistent with, for example, Eq. (8.128) or Eq. (22.24) of the textbook.

for $\mu = 0, 1, 2, 3$. In particular,

$$T_{11} = \frac{\omega}{\kappa} T_{01} = \frac{\omega^2}{\kappa^2} T_{00}. \tag{3.83}$$

Substituting these back into the Lagrangian in momentum space, we obtain

$$\begin{aligned} \mathcal{L} &= \frac{T'_{00} T_{00} \left(1 - 2\frac{\omega^2}{\kappa^2} + \frac{\omega^4}{\kappa^4}\right)}{2(\omega^2 - \kappa^2)} - \frac{T'_{02} T_{02} \left(1 - \frac{\omega^2}{\kappa^2}\right)}{\omega^2 - \kappa^2} - \frac{T'_{03} T_{03} \left(1 - \frac{\omega^2}{\kappa^2}\right)}{\omega^2 - \kappa^2} \\ &\quad + \frac{T'_{22} T_{22} + 2T'_{23} T_{23} + T'_{33} T_{33}}{2(\omega^2 - \kappa^2)} \\ &= \boxed{\frac{T'_{00} T_{00} \left(\omega^2 - \kappa^2\right)}{2\kappa^4} + \frac{T'_{02} T_{02}}{\kappa^2} + \frac{T'_{03} T_{03}}{\kappa^2} + \frac{T'_{22} T_{22} + 2T'_{23} T_{23} + T'_{33} T_{33}}{2(\omega^2 - \kappa^2)}}. \end{aligned} \tag{3.84}$$

We identify three apparent causally propagating degrees of freedom (i.e., non-instantaneous components):

$$\boxed{\mathcal{L}_{\text{causal}} = \frac{T'_{22} T_{22} + 2T'_{23} T_{23} + T'_{33} T_{33}}{2(\omega^2 - \kappa^2)}}. \tag{3.85}$$

(d) Since the graviton is massless, according to Wigner's classification, only two physical propagating degrees of freedom should remain. We can reduce the redundant one in Eq. (3.85) by adding another Lorentz-invariant term to the Lagrangian: $cT_{\mu\mu}\frac{1}{k^2}T_{\nu\nu}$. More concretely, we add

$$c(T'_{22} + T'_{33} + \cdots \frac{1}{k^2}(T_{22} + T_{33} + \cdots), \tag{3.86}$$

where $\cdots$ refers to non-causal terms, which we do not care about. The causal part of the Lagrangian now becomes

$$\mathcal{L}_{\text{causal}} = \frac{(1+2c)T'_{22} T_{22} + 2T'_{23} T_{23} + (1+2c)T'_{33} T_{33} + 2cT'_{22} T_{33} + 2cT'_{33} T_{22}}{2(\omega^2 - \kappa^2)}. \tag{3.87}$$

To eliminate the redundant degree of freedom, we need to set $c = -\frac{1}{4}$:

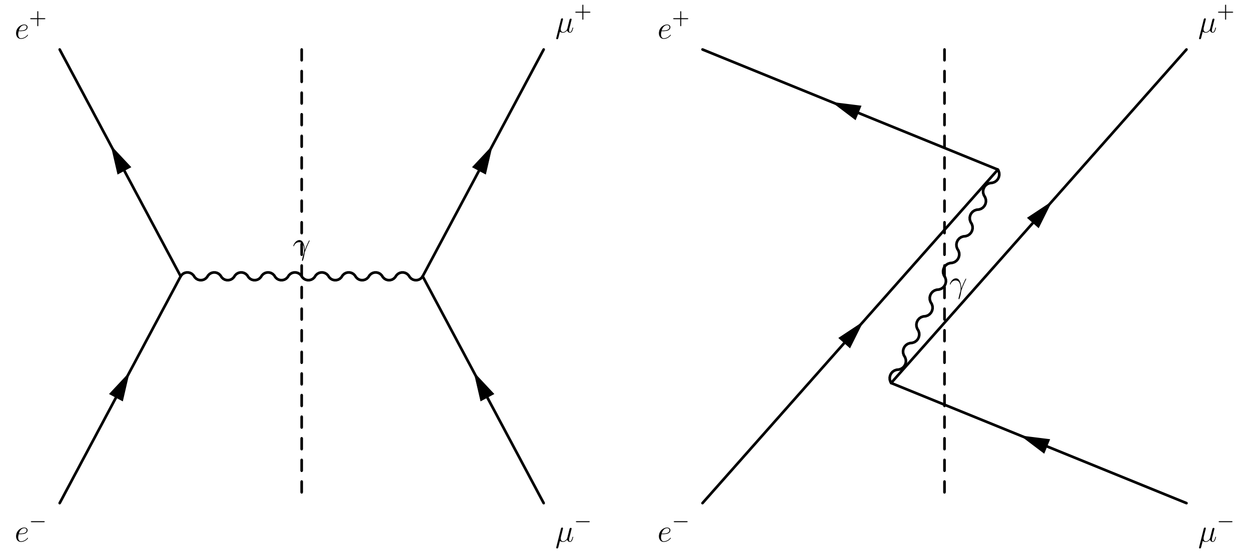
$$\begin{aligned} \mathcal{L}_{\text{causal}} &= \frac{\frac{1}{2}T'_{22} T_{22} + 2T'_{23} T_{23} + \frac{1}{2}T'_{33} T_{33} - \frac{1}{2}T'_{22} T_{33} - \frac{1}{2}T'_{33} T_{22}}{2(\omega^2 - \kappa^2)} \\ &= \boxed{\frac{\frac{1}{4}(T'_{22} - T'_{33})(T_{22} - T_{33}) + T'_{23} T_{23}}{(\omega^2 - \kappa^2)}}. \end{aligned} \tag{3.88}$$

Consequently, the two remaining causally propagating degrees of freedom are $\boxed{\frac{1}{2}(h_{22} - h_{33})}$ and $\boxed{h_{23}}$.

Chapter 4

Old-fashioned perturbation theory

4.1



(a) Retarded case

(b) Advanced case

Fig. 4.1: The vertical dashed line indicates the time at which the intermediate state is evaluate.

(a) The diagrams corresponding to the two possible time orderings for process $e^+e^- \to \gamma \to \mu^+\mu^-$ are shown in Fig. 4.1. For the retarded case, the intermediate photon travels forward in time. Therefore, the electron pair is annihilated before the muon pair is created. Hence, the intermediate state consists solely of an *on-shell* photon with energy

$$E_0^{(R)} = E_\gamma = |\vec{p}_\gamma| = |\vec{p}_1 + \vec{p}_2|. \tag{4.1}$$

For the advanced case, the intermediate photon travels backward in time, and the electron pair is annihilated after the muon pair is created. Hence, the energy of the intermediate state includes the electron pair, the muon pair, and the *on-shell* intermediate photon:

$$E_0^{(A)} = E_1 + E_2 + E_3 + E_4 + E_\gamma = E_1 + E_2 + E_3 + E_4 + |\vec{p}_1 + \vec{p}_2|. \tag{4.2}$$

In both cases, we have $E_i = E_1 + E_2 = E_f = E_3 + E_4$ and $E_\gamma = |\vec{p}_\gamma| = |\vec{p}_1 + \vec{p}_2| = |\vec{p}_3 + \vec{p}_4|$.

Therefore, for the retarded case,

$$\frac{1}{E_i - E_0^{(R)}} = \boxed{\frac{1}{E_1 + E_2 - E_\gamma}}, \tag{4.3}$$

and for the advanced case,

$$\frac{1}{E_i - E_0^{(A)}} = \boxed{\frac{1}{-E_3 - E_4 - E_\gamma}}. \tag{4.4}$$

(b) From part (a), we have

$$\begin{aligned} T_{fi}^{(R)} + T_{fi}^{(A)} &= \frac{e^2}{E_1 + E_2 - E_\gamma} + \frac{e^2}{-E_3 - E_4 - E_\gamma} \\ &= \frac{e^2}{(E_1 + E_2) - E_\gamma} - \frac{e^2}{(E_1 + E_2) + E_\gamma} \\ &= \boxed{\frac{2E_\gamma e^2}{E_i^2 - E_\gamma^2}}. \end{aligned} \tag{4.5}$$

Defining $k^\mu \equiv p_1^\mu + p_2^\mu = (E_1 + E_2, \vec{p}_1 + \vec{p}_2) = (E_i, \vec{p}_\gamma)$ as the 4-momentum of the virtual *off-shell* photon, we identify $k^2 = E_i^2 - |\vec{p}_\gamma|^2 = E_i^2 - |\vec{p}_1 + \vec{p}_2|^2 = E_i^2 - E_\gamma^2$ as precisely as the denominator above.

Chapter 5

Cross sections and decay rates

5.1

From the definition of LIPS (cf. Eq. (5.21) in the textbook), we have

$$\begin{aligned} d\Pi_{LIPS} &= (2\pi)^4\delta^4\left(\sum p\right)\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{d^3p_B}{(2\pi)^3}\frac{1}{2E_B} \\ &= \frac{1}{16\pi^2}d\Omega\int dp_f p_f^2\frac{1}{E_f}\frac{1}{E_B}\delta(E_f+E_B-E_i-m_A), \end{aligned} \tag{5.1}$$

where $p_f \equiv |\vec{p}_f|$, and note that $\vec{p}_B = \vec{p}_i - \vec{p}_f$, implicitly constrained by the integrated-out delta function. Therefore, $p_B^2 = p_i^2 + p_f^2 - 2p_ip_f\cos\theta$.

Now, define $x(p_f) = E_f + E_B - E_i - m_A$. Then,

$$\frac{dx}{dp_f} = \frac{dE_f}{dp_f} + \frac{dE_B}{dp_B}\frac{dp_B}{dp_i} = \frac{p_f}{E_f} + \frac{p_f - p_i\cos\theta}{E_B} = \frac{E_Bp_f + E_fp_f - E_fp_i\cos\theta}{E_BE_f}. \tag{5.2}$$

Substituting this expression back into the LIPS, we obtain

$$\begin{aligned} d\Pi_{LIPS} &= \frac{1}{16\pi^2}d\Omega\int_{m_f+E_B|_{p_f=0}-E_i-m_A}^{\infty} dx\left[p_f^2\frac{\delta(x)}{E_Bp_f+E_fp_f-E_Fp_i\cos\theta}\right] \\ &= \frac{1}{16\pi^2}d\Omega p_f\left[E_B+E_f\left(1-\frac{p_i}{p_f}\cos\theta\right)\right]^{-1}\theta(m_f+E_B|_{p_i=0}-E_i-m_A). \end{aligned} \tag{5.3}$$

Next, we plug this result into Eq. (5.22) of the textbook. Notice that $|\vec{v}_i - \vec{v}_A| = |\vec{v}_i| = \frac{p_i}{E_i}$. Therefore, we find

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{(2E_i)(2E_A)\frac{p_i}{E_i}}|\mathcal{M}|^2\frac{1}{16\pi^2}\left[E_B+E_f\left(1-\frac{p_i}{p_f}\cos\theta\right)\right]^{-1}p_f\theta(m_f+E_B|_{p_i=0}-E_i-m_A) \\ &= \frac{1}{64\pi^2m_A}\left[E_B+E_f\left(1-\frac{p_i}{p_f}\cos\theta\right)\right]^{-1}\frac{p_f}{p_i}|\mathcal{M}|^2\theta(m_f+E_B|_{p_i=0}-E_i-m_A). \end{aligned} \tag{5.4}$$

5.2

In Problem 2.6, I demonstrated that the integration measure $\int \frac{d^3k}{2\omega_k} = \int \frac{d^3k}{2E_k}$ is Lorentz invariant. Additionally, $\delta^4\left(\sum p\right)$ remains Lorentz invariant because 4-momentum conservation holds true in all inertial frames. Consequently, since $d\Pi_{\text{LIPS}}$ consists solely of Lorentz invariant factors, the entire expression is Lorentz invariant.

5.3

(a) We shall work in the rest frame of the decaying muon. Without loss of generality, let the outgoing electron-neutrino define the z-axis of our reference frame. Starting from the definition of LIPS (cf. Eq. (5.21) in the textbook), we have

$$d\Pi_{LIPS} = (2\pi)^4\delta^4\left(\sum p\right)\frac{d^3p_e}{(2\pi)^3}\frac{d^3p_{\bar{\nu}_e}}{(2\pi)^3}\frac{d^3p_{\nu_\mu}}{(2\pi)^3}\frac{1}{(2E)(2E_{\nu_\mu})(2E_e)} \tag{5.5}$$

Integrating over $\vec{p}_{\nu_\mu}$ via the δ-function, the spatial part fixes

$$\vec{p}_{\nu_\mu} = -(\vec{p}_e + \vec{p}_{\nu_e}). \tag{5.6}$$

Squaring both sides and using the approximation of massless electron and neutrinos, where $E_e = |\vec{p}_e|$ and $E_\nu = |\vec{p}_\nu|$, we can express E_{ν_μ} as

$$E_{\nu_\mu}^2 = E_e^2 + E^2 + 2EE_e\cos\theta, \tag{5.7}$$

with θ being the angle between the outgoing electron and the z-axis. Also,

$$\frac{dE_{\nu_\mu}}{d(\cos\theta)} = \frac{EE_e}{\sqrt{E_e^2 + E^2 + 2EE_e\cos\theta}} = \frac{EE_e}{E_{\nu_\mu}}. \tag{5.8}$$

Then,

$$\begin{aligned}
d\Pi_{LIPS} &= \frac{1}{8(2\pi)^5}\delta(m - E - E_e - E_{\nu_\mu})\frac{d^3p_e d^3p_{\bar{\nu}_e}}{EE_eE_{\nu_\mu}} \\
&= \frac{1}{4(2\pi)^4}dEE\frac{d^3p_e}{E_eE_{\nu_\mu}}\delta(m - E - E_e - E_{\nu_\mu}) \\
&= \frac{1}{4(2\pi)^3}dE\int dE_e\int_{-1}^{1}\frac{EE_e}{E_{\nu_\mu}}d(\cos\theta)\delta(m - E - E_e - E_{\nu_\mu}) \\
&= \frac{1}{4(2\pi)^3}dE\int_0^\infty dE_e\int_{|E-E_e|}^{E+E_e}dE_{\nu_\mu}\delta(m - E - E_e - E_{\nu_\mu}) \\
&= \frac{1}{4(2\pi)^3}\theta(m - 2E)dE\int_{m/2-E}^{m/2}dE_e \\
&= \frac{1}{4(2\pi)^3}\theta(m - 2E)EdE.
\end{aligned} \tag{5.9}$$

From Eq. (5.24) of the textbook, we get

$$\begin{aligned}\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) &= \frac{1}{2m}\int |\mathcal{M}|^2 d\Pi_{LIPS} \\ &= \frac{G_F^2}{2\pi^3}\int_0^\infty dE\ \theta(m-2E)(m^2-2mE)E^2 \\ &= \frac{G_F^2 m}{2\pi^3}\int_0^{m/2} dE(mE^2-2E^3) \\ &= \boxed{\frac{G_F^2 m^5}{192\pi^3}}\ .\end{aligned} \tag{5.10}$$

(b) Substituting the numerical values $G_F = 1.166\times10^{-5}$ GeV^{-2}, $m = 105.66$ MeV[1], we obtain

$$\tau_{\text{theory}} = \Gamma^{-1}_{\text{theory}} = \frac{192\pi^3}{(1.166\times10^{-5}\ \text{GeV}^{-2})^2(105.66\ \text{MeV})^5}\hbar \approx \boxed{2.19\ \mu\text{s}}\ . \tag{5.11}$$

This corresponds to a relative difference of

$$\frac{\tau_{\text{obs}}-\tau_{\text{theory}}}{\tau_{\text{theory}}} \approx \boxed{0.46\ \%}\ , \tag{5.12}$$

compared to the experimental measurement.

At first glance, one might suspect that this discrepancy arises from neglecting the electron mass in the derivation of the decay rate. However, according to Eq. (31.3) of the textbook, the leading tree-level correction due to the massive electron is of order $\mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right) \approx 2\times10^{-5}$, which is far too small to account for the observed deviation. Rather, this deviation is likely attributable to the radiative correction stemming from the interference between the tree-level diagram and the QED 1-loop diagram, which appears at order $\mathcal{O}(\alpha_e) \approx 0.01$, aligning well with expectations.

5.4

For circular polarization, we can take the photon polarization vectors produced by the incoming electrons (cf. Eq. (A.48) of the textbook) as

$$\begin{aligned}\epsilon_L^\mu &= \frac{1}{\sqrt{2}}(0,1,-i,0), \\ \epsilon_R^\mu &= \frac{1}{\sqrt{2}}(0,1,i,0).\end{aligned} \tag{5.13}$$

[1] It is important to emphasize that the decay rate scales with the 5th power of the mass, making it highly sensitive. Using $m = 106$ MeV as given in the textbook would artificially inflate the relative deviation by nearly an order of magnitude!

Note that the outgoing muon momenta in Eq. (5.48) result from a rotation in the y–z plane applied to the incoming electron momenta from Eq. (5.45). That is,

$$R_x(\theta)p_1^\mu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix} p_1^\mu = p_3^\mu,$$

and

$$R_x(\theta)p_2^\mu = p_4^\mu.$$

Applying the same y-z plane rotation $R_x(\theta)$ to the polarization vectors gives the circularly polarized states of the photon exchanged in the final state:

$$\begin{aligned} \bar{\epsilon}_L^\mu &= R_x(\theta)\epsilon_L = \frac{1}{\sqrt{2}}(0, 1, -i\cos\theta, i\sin\theta), \\ \bar{\epsilon}_R^\mu &= R_x(\theta)\epsilon_R = \frac{1}{\sqrt{2}}(0, 1, i\cos\theta, -i\sin\theta). \end{aligned} \tag{5.14}$$

It is straightforward to verify that these polarizations are orthogonal to the outgoing momenta:

$$p_3 \cdot \bar{\epsilon}_L = p_4 \cdot \bar{\epsilon}_L = p_3 \cdot \bar{\epsilon}_R = p_4 \cdot \bar{\epsilon}_R = 0,$$

and properly normalized:

$$\bar{\epsilon}_L^* \cdot \bar{\epsilon}_L^\mu = \bar{\epsilon}_R^* \cdot \bar{\epsilon}_R^\mu = 1.$$

Therefore, summing over the amplitudes squares gives

$$\begin{aligned} \sum_{\text{states}} |\mathcal{M}|^2 &= |\epsilon_L \cdot \bar{\epsilon}_L^*|^2 + |\epsilon_L \cdot \bar{\epsilon}_R^*|^2 + |\epsilon_R \cdot \bar{\epsilon}_L^*|^2 + |\epsilon_R \cdot \bar{\epsilon}_R^*|^2 \\ &= \frac{1}{4}[(1+\cos\theta)^2 + (1-\cos\theta)^2 + (1-\cos\theta)^2 + (1+\cos\theta)^2] \\ &= 1 + \cos\theta^2, \end{aligned} \tag{5.15}$$

as expected.

5.5

(a) The classical Rutherford scattering differential cross section is given by

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{Z_1^2 Z_2^2 \alpha^2}{16 E_{\mathrm{K}}^2 \sin^4 \frac{\theta}{2}}}, \tag{5.16}$$

where

- Z_1: the number of unit charges carried by the incident particle,
- Z_2: the number of unit charges carried by the stationary heavy nucleus,
- α: the fine structure constant,

- E_{K}: the initial non-relativistic kinetic energy of the incident particle,
- θ: the scattering angle.

The assumptions underlying Rutherford scattering are: (1) the process is non-relativistic and (2) the scattering is elastic, so the recoil of the heavy nucleus can be neglected.

(b)
- The charge factor becomes $\frac{e^4}{4\pi^2} \to \frac{Z_1^2 Z_2^2 e^4}{4\pi^2} = 4Z_1^2 Z_2^2 \alpha^2$.
- The momentum transfer is $\vec{k} = \vec{p}_i - \vec{p}_f$ such that

$$|\vec{k}|^2 = |\vec{p}_i|^2 + |\vec{p}_f|^2 - 2\vec{p}_i \cdot \vec{p}_f = 2p^2(1-\cos\theta) = 4p^2 \sin^2 \frac{\theta}{2},$$

  where we have used the fact that the scattering is elastic, so $|\vec{p}_i| = |\vec{p}_f| = p$.
- The non-relativistic kinetic energy is $E_K = \frac{p^2}{2m}$, hence $p^2 = 2mE_K$.

Under these replacements, Coulomb scattering reproduces the classical Rutherford scattering cross section:

$$\frac{e^4 m^2}{4\pi^2} \frac{1}{\vec{k}^4} \to \frac{Z_1^2 Z_2^2 \alpha^2}{16 E_{\mathrm{K}}^2 \sin^4 \frac{\theta}{2}} \tag{5.17}$$

Note that the above derivation is identical to the one in Section 13.4 of the textbook. However, I suspect that Eq. (13.79) and Eq. (13.80) in the textbook contain typos: the denominators with π^4 should likely read π^2.

(c) The Feynman diagram is shown in Fig. 5.1.

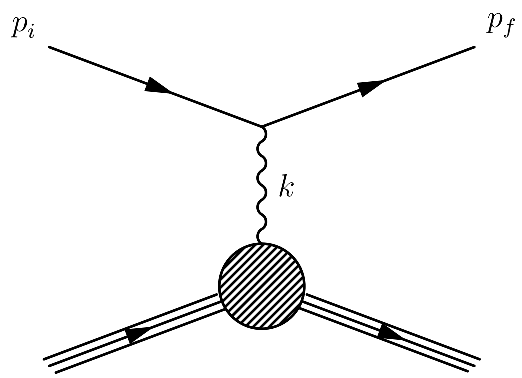


Fig. 5.1: The Feynman diagram for Rutherford scattering

Without loss of generality, assume the incoming α particle is moving along the z-axis, with four-momentum

$$p_i^\mu = (E, 0, 0, p), \tag{5.18}$$

and the outgoing α particle has four-momentum

$$p_f^\mu = (E, 0, p\sin\theta, p\cos\theta), \tag{5.19}$$

where we again used the fact that the scattering is elastic, so $E_i = E_f = E$, and azimuthal symmetry allows us to set $p_f^x = 0$.

The momentum of the virtual photon k^μ is then given by

$$\boxed{k^\mu = p_i^\mu - p_f^\mu = (0, 0, -p\sin\theta, p(1-\cos\theta))}\,, \tag{5.20}$$

where $p = \sqrt{E^2 - m_\alpha^2}$.

(d) This was already addressed in part (b).

(e) The tree-level result of QFT and the classical field description generally coincide. This can be understood via the path integral (cf. Eq. (14.31) of the textbook):

$$\langle 0; t_f | 0; t_i \rangle = N \int \mathcal{D}\Phi(\vec{x}, t) e^{\frac{i}{\hbar} S[\Phi]}, \tag{5.21}$$

where S is the action of the system, which has the same form in both classical and quantum contexts (interpreting Φ as a classical or quantum field). In the classical limit $\hbar \to 0$, the integral is dominated by the stationary point of the action, $\delta S = 0$, which yields the Euler-Lagrange equations. In QFT, this corresponds to the tree-level approximation. Thus, both frameworks agree at tree level.

(f) Møller scattering ($e^- e^- \to e^- e^-$) was originally derived under non-relativistic assumptions, without invoking QED. However, recall that in the derivation of the Coulomb scattering, one assumes the incident particle's mass is much smaller than that of the target (e.g., $m_e \ll m_p$), which is clearly not applicable for the Møller scattering. Furthermore, unlike in Coulomb scattering ($e^- p^+$ in the final state), the outgoing particles in Møller scattering ($e^- e^-$ in the final state) are indistinguishable, necessitating the inclusion of both t- and u-channel diagrams instead of a single t-channel diagram as in the Coulomb scattering.

Therefore, Eq. (5.41) of the textbook does not apply to Møller scattering.

5.6

I shall denote $p_1 \equiv p_{e^-}$, $p_2 \equiv p_{e^+}$, $p_3 \equiv p_{\mu^-}$, $p_4 \equiv p_{\mu^+}$.

(a) Using Eq. (5.45) and Eq. (5.48) from the textbook, we have

$$s = (p_1 + p_2)^2 = (E + E)^2 - (E - E)^2 = 4E^2 = \boxed{E_{\text{CM}}^2}\,. \tag{5.22}$$

$$t = (p_1 - p_3)^2 = (E - E)^2 - (0 - E\sin\theta)^2 - (E - E\cos\theta)^2 = \boxed{-2E^2(1-\cos\theta)}\,. \tag{5.23}$$

$$u = (p_1 - p_4)^2 = (E - E)^2 - (0 + E\sin\theta)^2 - (E + E\cos\theta)^2 = \boxed{-2E^2(1+\cos\theta)}\,. \tag{5.24}$$

(b) Define a shorthand $p_{ij} \equiv p_i^\mu p_j^\mu = p_i \cdot p_j$. Then,

$$\begin{aligned} s + t + u &= p_1^2 + p_2^2 + 2p_{12} + p_1^2 + p_3^2 - 2p_{13} + p_1^2 + p_4^2 - 2p_{14} \\ &= 2p_1^\mu(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu) + p_1^2 + p_2^2 + p_3^2 + p_4^2 \\ &= \sum_i p_i^2 \\ &= \boxed{\sum_i m_i^2}\,, \end{aligned} \tag{5.25}$$

where i runs over all the particle participating into the interactions, and we used the momentum conservation such that $p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu = 0$. If we take the ultra-relativistic limit that $m_e \to 0$ and $m_\mu \to 0$, then

$$s + t + u = 0. \tag{5.26}$$

(c) Note that $t^2 + u^2 = 4E^4[(1-\cos\theta)^2 + (1+\cos\theta)^2] = 8E^4(1+\cos^2\theta) = \frac{s^2}{2}(1+\cos^2\theta)$. Then, we can write

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{\text{CM}}^2}(1+\cos^2\theta) = \boxed{\frac{e^4}{32\pi^2 s^3}(t^2+u^2)}\,. \tag{5.27}$$

(d) We have already derived the general result in part (b):

$$s + t + u = \sum_i m_i^2 = 2m_e^2 + 2m_\mu^2. \tag{5.28}$$

Chapter 6

The S-matrix and time-ordered products

6.1

Starting from the Feynman propagator in Eq. (6.34) of the textbook,

$$
\begin{aligned}
D_F(x_1, x_2) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(x_1 - x_2)} \\
&= \int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} e^{ik(x_1 - x_2)} e^{is(k^2 - m^2)} e^{-\varepsilon s} \\
&= \int_0^\infty ds e^{-ism^2} \int \frac{d^4k}{(2\pi)^4} e^{i\left[k^2 s + k(x_1 - x_2)\right]},
\end{aligned}
\tag{6.1}
$$

where I have employed the Schwinger parameters $\frac{i}{A} = \int_0^\infty ds e^{isA}$ (cf. Eq. (B.5) of the textbook), which holds for $\mathrm{Im}(A) > 0$. Also, I have taken the limit $\lim_{\varepsilon \to 0} e^{-\varepsilon s} \to 1$ in the last line.

The d^4k integral is Gaussian and can be evaluated using Eq. (14.7) of the textbook:

$$
\int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2}\vec{p}^\dagger \mathbf{A} \vec{p} + \vec{J}^\dagger \vec{p}} = \sqrt{\frac{(2\pi)^n}{\det \mathbf{A}}} e^{\frac{1}{2}\vec{J}^\dagger \mathbf{A}^{-1} \vec{J}}
\tag{6.2}
$$

with $\mathbf{A} = -2isg^{\mu\nu}$ and $J^\mu = i(x_1^\mu - x_2^\mu)$. Note that $\det \mathbf{A} = -16s^4$ and $\mathbf{A}^{-1} = \frac{i}{2s} g_{\mu\nu}$. Thus, we have

$$
D_F(x_1, x_2) = \frac{-i}{16\pi^2} \int_0^\infty \frac{ds}{s^2} \exp\left[-i\left[\frac{(x_1 - x_2)^2}{4s} + sm^2\right]\right].
\tag{6.3}
$$

Taking the $m \to 0$ limit and defining $\beta \equiv \frac{1}{s}$, we obtain

$$\begin{aligned} D_F(x_1, x_2) &= \frac{-i}{16\pi^2} \int_0^\infty d\beta \exp\left[-i\frac{(x_1-x_2)^2}{4}\beta\right] \\ &= \frac{-i}{16\pi^2} \int_0^\infty d\beta \exp\left[-i\frac{(x_1-x_2)^2 - i\varepsilon}{4}\beta\right] \\ &= \frac{-i}{16\pi^2} \frac{-4i}{(x_1-x_2)^2 - i\varepsilon} \\ &= \boxed{-\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2 - i\varepsilon}}\,, \end{aligned} \tag{6.4}$$

where I inserted an $-i\varepsilon$ inside the exponent in the second line to ensure convergence of the integral for all values of $(x_1 - x_2)^2$. Note that the sign of $i\varepsilon$ must be negative; choosing $+i\varepsilon$ instead is not an innocuous distortion because it would lead to divergence in $e^{\varepsilon\beta} \to \infty$ as $\beta \to \infty$.

Side Remark: the physical role of this $i\varepsilon$ term differs from that appearing in the momentum-space Feynman propagator $\frac{i}{k^2-m^2+i\varepsilon}$; in particular, their mass dimensions are not even the same.

6.2

Starting from Eq. (6.26) of the textbook, one might naively expect the term

$$\langle 0|\phi_0(x_2)\phi_0(x_1)|0\rangle\theta(-\tau)$$

to correspond to the advanced propagator, and similarly,

$$\langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle\theta(\tau)$$

to correspond to the retarded propagator. However, these identifications can not be correct because they are not Lorentz invariant. For example, the term $\langle 0|\,\phi_0(x_1)\phi_0(x_2)\,|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik(x_1-x_2)}$ is Lorentz invariant (cf. Problem 2.6 and the fact that the exponential involves only Lorentz-invariant quantity). In contrast, the step function $\theta(t_2 - t_1) \equiv \theta(-\tau)$ is not invariant under Lorentz transformations. Hence, the whole term can not be Lorentz invariant.

To further illustrate this point, consider $\phi_0(x_1)$ and $\phi_0(x_2)$ are separated by a spacelike separation $(x_1 - x_2)^2 < 0$. Then, there exists a continuous Lorentz transformation that can reverse the time ordering, rendering $\theta(-\tau)$ ambiguous. Thus, any objects built from the product of $\theta(\pm\tau)$ with a Lorentz invariant function will fail to be invariant unless it vanishes in the spacelike region. Therefore, to construct a Lorentz invariant advanced (or retarded) propagator, it is insufficient to just enforce time ordering; the propagator must also vanish for spacelike separations.

To construct the correct form of the advanced propagator, we again start from Eq. (6.26) of the textbook, but reverse the time ordering in the first term by redefining $\tau \to -\tau$. Note that the exponential must remain unchanged—we must not allow both terms to carry either both positive or both negative frequency modes, as that would lead to a unphysical negative energy state. Accordingly, we must also redefine $\omega_k \to -\omega_k$ in the first term (*yep, this is exactly the Feynman–Stueckelberg interpretation*), so that $-i(-\omega_k)(-\tau) = -i\omega_k\tau$. This leads us to

$$D_A(x_1, x_2) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[-e^{-ik(x_1-x_2)} + e^{ik(x_1-x_2)}\right] \theta(-\tau). \tag{6.5}$$

This ensures that both terms now respect the correct time ordering $\theta(-\tau)$. The minus sign in front of the first term arises from the sign flip of ω_k. We now need to demonstrate it is indeed Lorentz invariant.

To save notations, define

$$D(x_1, x_2) \equiv \langle 0|\phi_0(x_1)\phi_0(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik(x_1-x_2)}, \tag{6.6}$$

which is Lorentz invariant (cf. Problem 2.6 and the fact that the exponential involves only Lorentz-invariant quantity). Then, the advanced propagator can be written as

$$D_A(x_1, x_2) = -\left[D(x_1, x_2) - D(x_2, x_1)\right] \theta(-\tau) = -\langle 0|[\phi_0(x_1), \phi_0(x_2)]|0\rangle\theta(-\tau) \tag{6.7}$$

Again, if the separation between x_1 and x_2 is spacelike, i.e., $(x_1 - x_2)^2 < 0$, then a continuous Lorentz transformation exists that can reverse the time ordering between the two events and the step function $\theta(-\tau)$ becomes frame-dependent and thus non-invariant. However, now, the commutator $\langle 0|[\phi_0(x_1), \phi_0(x_2)]|0\rangle = 0$ for spacelike separation, as it has support only on and inside the lightcone (cf. Eqs. (12.76)–(12.80) of the textbook), which is just the requirement of causality. Therefore, any ambiguity in $\theta(-\tau)$ is removed by the vanishing of the commutator.

Conversely, when $(x_1-x_2)^2 \geq 0$, i.e., when the points are separated by a timelike or lightlike interval, no continuous Lorentz transformation can invert their causal order. In this case, the step function $\theta(-\tau)$ is well-defined, and we conclude that the advanced propagator takes the form:

$$D_A(x_1, x_2) = \begin{cases} -\left[D(x_1, x_2) - D(x_2, x_1)\right] \theta(-\tau) & (x_1 - x_2)^2 \geq 0 \\ 0 & (x_1 - x_2)^2 < 0 \end{cases} \tag{6.8}$$

Since $D(x_1, x_2)$ is manifestly Lorentz invariant, and the use of $\theta(-\tau)$ is restricted to the causal regime where it is unambiguous, the advanced propagator D_A, defined in Eq. (6.5), is Lorentz invariant and has correct time ordering.

Similarly, the retarded propagator can be expressed as

$$D_R(x_1, x_2) = \left[D(x_1, x_2) - D(x_2, x_1)\right] \theta(\tau) = \langle 0|[\phi_0(x_1), \phi_0(x_2)]|0\rangle\theta(\tau), \tag{6.9}$$

where the time ordering is reversed in the second term of Eq. (6.26) of the textbook, along with the appropriate sign change of ω_k.

Continuing with the derivation,

$$\begin{aligned} D_A &= -\left[D(x_1, x_2) - D(x_2, x_1)\right] \theta(-\tau) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[-e^{i\vec{k}(\vec{x}_1-\vec{x}_2)} e^{-i\omega_k\tau} + e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)} e^{i\omega_k\tau}\right] \theta(-\tau) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)} \left[e^{i\omega_k\tau} - e^{-i\omega_k\tau}\right] \theta(-\tau), \end{aligned} \tag{6.10}$$

where we used the substitution $\vec{k} \to -\vec{k}$ in the first term, which leaves the integral measure invariant.

For $\tau < 0$, we close the contour in the lower half-plane. The relevant contour integrals are

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega - (\omega_k - i\varepsilon)} e^{i\omega\tau} = -2\pi i e^{i\omega_k \tau} \theta(-\tau) + \mathcal{O}(\varepsilon), \tag{6.11}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega - (-\omega_k - i\varepsilon)} e^{i\omega\tau} = -2\pi i e^{-i\omega_k \tau} \theta(-\tau) + \mathcal{O}(\varepsilon), \tag{6.12}$$

where the minus signs arise from clockwise contour closure.

Therefore, the advanced propagator becomes

$$\begin{aligned} D_A(x_1, x_2) &= \lim_{\varepsilon \to 0^-} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} \frac{i}{2\pi} \int d\omega e^{i\omega_k \tau} \left[\frac{1}{\omega - (\omega_k - i\varepsilon)} - \frac{1}{\omega - (-\omega_k - i\varepsilon)} \right] \\ &= \lim_{\varepsilon \to 0^-} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} \frac{i}{2\pi} \int d\omega e^{i\omega_k \tau} \left[\frac{2\omega_k}{(\omega + i\varepsilon)^2 - \omega_k^2} \right] \\ &= \lim_{\varepsilon \to 0^-} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k_0 + i\varepsilon)^2 - \vec{k}^2 - m^2} e^{ik(x_1 - x_2)} \\ &= \boxed{\lim_{\varepsilon \to 0^-} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(x_1 - x_2)}} . \end{aligned} \tag{6.13}$$

For the retarded propagator, we close the contour in the upper half-plane since $\tau > 0$, obtaining

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega - (\omega_k + i\varepsilon)} e^{i\omega\tau} = 2\pi i e^{i\omega_k \tau} \theta(\tau) + \mathcal{O}(\varepsilon), \tag{6.14}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega - (-\omega_k + i\varepsilon)} e^{i\omega\tau} = 2\pi i e^{-i\omega_k \tau} \theta(\tau) + \mathcal{O}(\varepsilon). \tag{6.15}$$

Hence,

$$\begin{aligned} D_R(x_1, x_2) &= \lim_{\varepsilon \to 0^+} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} \left[-e^{i\omega_k \tau} + e^{-i\omega_k \tau} \right] \theta(\tau) \\ &= \lim_{\varepsilon \to 0^+} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} \frac{i}{2\pi} \int d\omega e^{i\omega_k \tau} \left[\frac{1}{\omega - (\omega_k + i\varepsilon)} - \frac{1}{\omega - (-\omega_k + i\varepsilon)} \right] \\ &= \lim_{\varepsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k_0 - i\varepsilon)^2 - \vec{k}^2 - m^2} e^{ik(x_1 - x_2)} \\ &= \boxed{\lim_{\varepsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 - i\varepsilon} e^{ik(x_1 - x_2)}} . \end{aligned} \tag{6.16}$$

Side Remark: By the way, there exists a very interesting relationship connecting the advanced and retarded propagators to the Feynman propagator. For the advanced (retarded) propagator, both poles of the integrand lie below (above) the real axis: $k_0 = \pm\omega_k - i\varepsilon$ (resp. $k_0 = \pm\omega_k + i\varepsilon$). In contrast, for the Feynman propagator, the poles are located on opposite sides of the real axis: $k_0 = \pm\omega_k \mp i\varepsilon$. These differences in pole prescription lead to the following identities in momentum space:

$$\begin{aligned}
\Pi_A(k) &= \frac{i}{2\omega_k}\left[\frac{1}{\omega-(\omega_k-i\varepsilon)} - \frac{1}{\omega-(-\omega_k-i\varepsilon)}\right] \\
&= \frac{i}{2\omega_k}\left[\frac{1}{\omega-\omega_k+i\varepsilon} - \frac{1}{\omega+\omega_k-i\varepsilon} - \frac{1}{\omega+\omega_k+i\varepsilon} + \frac{1}{\omega+\omega_k-i\varepsilon}\right] \\
&= \Pi_F(k) - \frac{i}{2\omega_k}\left[\frac{1}{\omega+\omega_k+i\varepsilon} - \frac{1}{\omega+\omega_k-i\varepsilon}\right] \\
&= \Pi_F(k) + \frac{1}{\omega_k}\,\mathrm{Im}\left[\frac{1}{\omega+\omega_k+i\varepsilon}\right] \\
&= \Pi_F(k) - \frac{\pi}{\omega_k}\delta(\omega+\omega_k),
\end{aligned} \tag{6.17}$$

where I have used Eq. (24.26) of the textbook:

$$\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0-\omega_k-i\varepsilon} = -2\pi i\delta(k_0-\omega_k). \tag{6.18}$$

Analogously, for the retarded propagator:

$$\begin{aligned}
\Pi_R(k) &= \frac{i}{2\omega_k}\left[\frac{1}{\omega-(\omega_k+i\varepsilon)} - \frac{1}{\omega-(-\omega_k+i\varepsilon)}\right] \\
&= \Pi_F(k) + \frac{1}{\omega_k}\,\mathrm{Im}\left[\frac{1}{\omega-\omega_k+i\varepsilon}\right] \\
&= \Pi_F(k) - \frac{\pi}{\omega_k}\delta(\omega-\omega_k).
\end{aligned} \tag{6.19}$$

As a spoiler, I just verified Eq. (24.27) of the textbook[a]. These propagators are closely related to the optical theorem and the Cutkosky cutting rules.

[a] Except that I used a flipped convention of "advanced" and "retarded", which is just a matter of "whose point of view" with nothing physical.

6.3

Starting from the general basis for a state with n particles, we define

$$|\psi_n\rangle = \mathcal{N} a^\dagger_{k_1}\cdots a^\dagger_{k_n}\,|\Omega\rangle\,, \tag{6.20}$$

where $\mathcal{N}$ is a normalization factor. The normalization condition for the vacuum state imposes

$$1 = \langle\Omega|\Omega\rangle = |\mathcal{N}|^2. \tag{6.21}$$

We proceed by examining the matrix element of an operator $\mathcal{O}$, sandwiched between basis states as $\langle\psi_n|\,\mathcal{O}\,|\psi_m\rangle$. Since we are concerned only with non-trivial scattering processes, we assume $|\psi_m\rangle \neq |\psi_n\rangle$ — that is, either $m \neq n$ or, if $m = n$, at least one momentum in the final state differs from that in the initial state.

Furthermore, we impose that $\mathcal{O}$ must act non-trivially on all m particles in the initial state and all n particles in the final state. Without this condition, subsets of particles could remain unaffected, resulting in trivial identity factors, which can be factored out from the scattering matrix[1].

The matrix element is then given by

$$\begin{aligned}\langle\psi_n|\,\mathcal{O}\,|\psi_m\rangle = |\mathcal{N}|^2 \sum_{k,l} \int dq_1 \cdots dq_k dp_1 \cdots dp_l C_{kl}(q_1, \ldots, p_l) \\ \times \langle\Omega|\, a_{k_n} \cdots a_{k_1} a^\dagger_{q_1} \cdots a^\dagger_{q_k} a_{p_l} \cdots a_{p_1} a^\dagger_{k'_1} \cdots a^\dagger_{k'_m} \,|\Omega\rangle\,.\end{aligned} \tag{6.22}$$

Note that the non-triviality condition implies that only terms with $k \geq n$ and $l \geq m$ can be non-trivial in the sum.

Now, we can systematically commute all annihilation operators to the right and all creation operators to the left. Each time an annihilation operator passes a creation operator, it produces a Dirac delta function $\delta(p-k)$ through contraction. For instance, we commute the group $a^\dagger_{q_1} \cdots a^\dagger_{q_k}$ past $a_{k_n} \cdots a_{k_1}$ to the left. If any of these creation operators remains uncontracted and acts directly on the vacuum in the final state, the matrix element vanishes. Therefore, to obtain a nonzero result, it must be that $k \leq n$. The same argument also leads to $l \leq m$.

Combining these with the non-trivial scattering conditions, we conclude that only term with $k = n$ and $l = m$ survives. After performing the contractions, the surviving term produces a series of delta functions. Note this procedure is de facto "*normal ordering*". By normal ordering these creation/annihilation operators, the only parts that are not vanishing in vacuum matrix elements are all kinds of possible contractions resulting delta functions. One can refer to Section 7.A of the textbook for further details of normal ordering and contractions. We can then carry out the integral:

$$\begin{aligned}\langle\psi_n|\,\mathcal{O}\,|\psi_m\rangle &= \int dq_1 \cdots dq_k dp_1 \cdots dp_l C_{kl}(q_1, \ldots, p_l) \left[\delta(q_1 - k_1) \cdots \delta(p_1 - k'_1) + (\text{permutations})\right] \\ &= n!m!C_{nm}(k_1, \cdots, k'_m).\end{aligned} \tag{6.23}$$

Here, the factors $n!m!$ account for the number of ways to contract the creation and annihilation operators: there are $n!$ ways to contract $a_{k_n} \cdots a_{k_1}$ with $a^\dagger_{q_1} \cdots a^\dagger_{q_n}$ and $m!$ ways to contract $a_{p_m} \cdots a_{p_1}$ with $a^\dagger_{k'_1} \cdots a^\dagger_{k'_m}$. These combinatorial factors are often absorbed into the definition of $\mathcal{O}$ itself, effectively redefining $\mathcal{O} \to \frac{1}{n!m!}\mathcal{O}$ where it appears in the Lagrangian.

[1] More formally, we need to invoke the cluster decomposition principle here, and the non-interacting part of the scattering amplitude corresponds to the disconnected pieces of Green's functions.

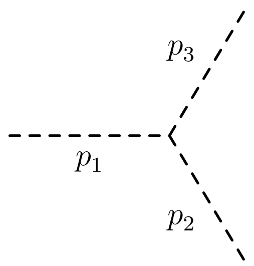
Chapter 7

Feynman rules

7.1

(a)

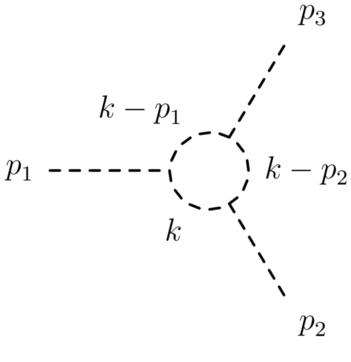
$$i\mathcal{M}_{\text{tree}} = \boxed{\quad}$$



$$= ig \; . \tag{7.1}$$

(b)

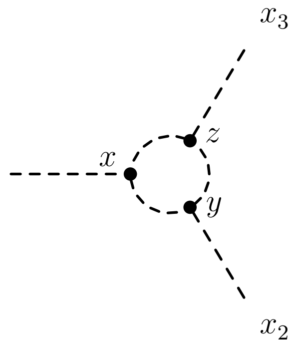
$$i\mathcal{M}_{\text{1-loop}} =$$



$$= \boxed{(ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k-p_2)^2 - m^2 + i\varepsilon} \frac{i}{(k-p_1)^2 - m^2 + i\varepsilon}} \; . \tag{7.2}$$

(c)

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle \quad = \quad x_1$$



$$
\begin{aligned}
&= \boxed{(ig)^3 \int d^4x \int d^4y \int d^4z D(x_1,x)D(x,y)D(y,x_2)D(y,z)D(z,x_3)D(z,x)} \\
&= (ig)^3 \int d^4x \int d^4y \int d^4z \int \frac{d^4p_1'}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4p_2'}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4p_3'}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \\
&\quad \times e^{ip_1'(x_1-x)} e^{ik_1(x-y)} e^{ip_2'(y-x_2)} e^{ik_2(y-z)} e^{ip_3'(z-x_3)} e^{ik_3(z-x)} \\
&\qquad \times \frac{i}{p_1'^2-m^2+i\epsilon} \frac{i}{k_1^2-m^2+i\epsilon} \frac{i}{p_2'^2-m^2+i\epsilon} \frac{i}{k_2^2-m^2+i\epsilon} \frac{i}{p_3'^2-m^2+i\epsilon} \frac{i}{k_3^2-m^2+i\epsilon}.
\end{aligned}
\tag{7.3}
$$

(d) Applying LSZ formula (cf. Eq. (6.1) of the textbook) to Eq. (7.3),

$$
\begin{aligned}
\langle f|S|i\rangle &= \left[i\int d^4x_1 e^{-ip_1x_1}(\Box+m^2)\right]\left[i\int d^4x_2 e^{ip_2x_2}(\Box+m^2)\right]\left[i\int d^4x_3 e^{ip_3x_3}(\Box+m^2)\right] \\
&\qquad \times \langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle \\
&= \left[-i\int d^4x_1 e^{-ip_1x_1}(p_1^2-m^2)\right]\left[-i\int d^4x_2 e^{ip_2x_2}(p_2^2-m^2)\right]\left[-i\int d^4x_3 e^{ip_3x_3}(p_3^2-m^2)\right] \\
&\qquad \times \langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle \\
&= \int \frac{d^4p_1'}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4p_2'}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4p_3'}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x \int d^4y \int d^4z \\
&\quad \times e^{i(-p_1+p_1')x_1} e^{i(p_2-p_2')x_2} e^{i(p_3-p_3')x_3} e^{i(-p_1'+k_1-k_3)x} e^{i(-k_1+p_2'+k_2)y} e^{i(-k_2+p_3'+k_3)z} \\
&\qquad \times (ig)^3 \frac{p_1^2-m^2}{p_1'^2-m^2+i\epsilon} \frac{i}{k_1^2-m^2+i\epsilon} \frac{p_2^2-m^2}{p_2'^2-m^2+i\epsilon} \frac{i}{k_2^2-m^2+i\epsilon} \frac{p_3^2-m^2}{p_3'^2-m^2+i\epsilon} \frac{i}{k_3^2-m^2+i\epsilon} \\
&= (ig)^3 \int \frac{d^4p_1'}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4p_2'}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4p_3'}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} (2\pi)^{24} \\
&\quad \times \delta^4(-p_1+p_1')\delta^4(p_2-p_2')\delta^4(p_3-p_3')\delta^4(-p_1'+k_1-k_3)\delta^4(-k_1+p_2'+k_2)\delta^4(-k_2+p_3'+k_3) \\
&\qquad \times \frac{p_1^2-m^2}{p_1'^2-m^2+i\epsilon} \frac{i}{k_1^2-m^2+i\epsilon} \frac{p_2^2-m^2}{p_2'^2-m^2+i\epsilon} \frac{i}{k_2^2-m^2+i\epsilon} \frac{p_3^2-m^2}{p_3'^2-m^2+i\epsilon} \frac{i}{k_3^2-m^2+i\epsilon} \\
&= \boxed{(2\pi)^4\delta^4(p_1-p_2-p_3)(ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k-p_2)^2-m^2+i\epsilon} \frac{i}{(k-p_1)^2-m^2+i\epsilon}},
\end{aligned}
\tag{7.4}
$$

where I relabeled $k_1 \to k$ in the last line. This expression matches exactly with Eq. (7.2) from part (b), up to the overall factor $(2\pi)^4\delta^4(p_1 - p_2 - p_3)$ that enforces momentum conservation and factors out of the matrix element.

7.2

$$i\mathcal{M}_{6pt} = i\lambda. \tag{7.5}$$

$$i\mathcal{M}_{3pt} = ig. \tag{7.6}$$

The connected $2 \to 4$ diagram with a 6-point vertex contributes to the S-matrix as

$$\langle f|S|i\rangle = (2\pi)^4\delta^4\left(\sum p\right) i\mathcal{M}_{6pt} = (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)i\lambda, \tag{7.7}$$

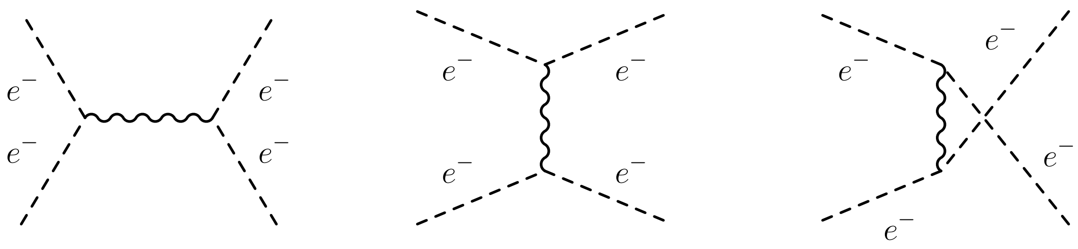
while the disconnected $2 \to 4$ diagram composed of two $1 \to 3$ subdiagrams with 3-point vertices contributes to the S-matrix as the square of the $1 \to 3$ amplitude,

$$\begin{aligned} \langle f|S|i\rangle &= -(2\pi)^8\delta^4\left(\sum_{\text{subset}_1} p_m\right)\delta^4\left(\sum_{\text{subset}_2} p_n\right)(\mathcal{M}_{3pt})^2 \\ &= -(2\pi)^8\delta^4(p_1 - p_3 - p_4)\delta^4(p_2 - p_5 - p_6)g^2 + \text{ permutations of final states }. \end{aligned} \tag{7.8}$$

To check whether there is any interference between the connected and disconnected diagrams, one could sum the two and then take the square. However, it is obvious that each additional delta function from a disconnected piece introduces an extra factor of spacetime volume $\frac{TV}{(2\pi)^4}$, which is formally infinite. Hence, disconnected diagrams are always infinitely larger than the connected ones, and any interference term vanishes.

7.3

(a) The diagrams are shown in Fig. 7.1.



(a) s-channel (b) t-channel (c) u-channel

Fig. 7.1: Spinless non-relativistic Møller scattering $e^-e^- \to e^-e^-$

(b) The s-channel diagram Fig. 7.1a is forbidden due to charge conservation in real QED.

(c)

$$i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u = (iem_e)\frac{i}{t}(iem_e) - (iem_e)\frac{i}{u}(iem_e) = \boxed{-ie^2m_e^2\left[\frac{1}{t} - \frac{1}{u}\right]}. \tag{7.9}$$

(d) Since spin is conserved at each vertex, the allowed spin combinations are:

- ***t*-channel**: (i) $|\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle$, (ii) $|\downarrow\downarrow\rangle \to |\downarrow\downarrow\rangle$, (iii) $|\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle$, (iv) $|\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle$.
- ***u*-channel**: (i) $|\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle$, (ii) $|\downarrow\downarrow\rangle \to |\downarrow\downarrow\rangle$, (iii) $|\uparrow\downarrow\rangle \to |\downarrow\uparrow\rangle$, (iv) $|\downarrow\uparrow\rangle \to |\uparrow\downarrow\rangle$.

Then,

$$i\mathcal{M}_{|\uparrow\uparrow\rangle\to|\uparrow\uparrow\rangle} = i\mathcal{M}_{|\downarrow\downarrow\rangle\to|\downarrow\downarrow\rangle} = -ie^2 m_e^2 \left(\frac{1}{t} - \frac{1}{u}\right), \tag{7.10}$$

$$i\mathcal{M}_{|\uparrow\downarrow\rangle\to|\uparrow\downarrow\rangle} = i\mathcal{M}_{|\downarrow\uparrow\rangle\to|\downarrow\uparrow\rangle} = -\frac{ie^2 m_e^2}{t}, \tag{7.11}$$

$$i\mathcal{M}_{|\uparrow\downarrow\rangle\to|\downarrow\uparrow\rangle} = i\mathcal{M}_{|\downarrow\uparrow\rangle\to|\uparrow\downarrow\rangle} = \frac{ie^2 m_e^2}{u}. \tag{7.12}$$

(e) The squared amplitudes are:

$$|\mathcal{M}_{|\uparrow\uparrow\rangle\to|\uparrow\uparrow\rangle}|^2 = |\mathcal{M}_{|\downarrow\downarrow\rangle\to|\downarrow\downarrow\rangle}|^2 = e^4 m_e^4 \left(\frac{1}{t^2} + \frac{1}{u^2} - \frac{2}{tu}\right), \tag{7.13}$$

$$|\mathcal{M}_{|\uparrow\downarrow\rangle\to|\uparrow\downarrow\rangle}|^2 = |\mathcal{M}_{|\downarrow\uparrow\rangle\to|\downarrow\uparrow\rangle}|^2 = \frac{e^4 m_e^4}{t^2}, \tag{7.14}$$

$$|\mathcal{M}_{|\uparrow\downarrow\rangle\to|\downarrow\uparrow\rangle}|^2 = |\mathcal{M}_{|\downarrow\uparrow\rangle\to|\uparrow\downarrow\rangle}|^2 = \frac{e^4 m_e^4}{u^2}. \tag{7.15}$$

Let $p_1 = (E, \vec{p}_i)$, $p_2 = (E, -\vec{p}_i)$ and $p_3 = (E, \vec{p}_f)$, $p_4 = (E, -\vec{p}_f)$ be the initial and final four-momenta in the CM frame, with $E = \frac{1}{2}E_{\text{CM}}$. Then,

$$t = (p_1 - p_3)^2 = -2p^2(1 - \cos\theta), \tag{7.16}$$

$$u = (p_1 - p_4)^2 = -2p^2(1 + \cos\theta), \tag{7.17}$$

where θ is the scattering angle, and $p = |\vec{p}_i| = |\vec{p}_f| = \sqrt{E^2 - m_e^2} = \frac{1}{2}\sqrt{E_{\text{CM}}^2 - 4m_e^2}$.

The CM differential cross section is given by (cf. Eq. (5.33) of the textbook):

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{64\pi^2 E_{\text{CM}}^2} \times \frac{1}{2} \times \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{e^4 m_e^4}{128\pi^2 E_{\text{CM}}^2} \left(\frac{1}{t^2} + \frac{1}{u^2} - \frac{1}{tu}\right) \\
&= \frac{e^4 m_e^4}{512\pi^2 E_{\text{CM}}^2 p^4} \left[\frac{1}{(1-\cos\theta)^2} + \frac{1}{(1+\cos\theta)^2} - \frac{1}{1-\cos^2\theta}\right] \\
&= \frac{e^4 m_e^4}{32\pi^2 E_{\text{CM}}^2 (E_{\text{CM}}^2 - 4m_e^2)^2} \frac{1 + 3\cos^2\theta}{\sin^4\theta},
\end{aligned} \tag{7.18}$$

where the factor of $\frac{1}{2}$ in the first line accounts for the two identical particles in the final state, and the factor of $\frac{1}{4}$ averages out our ignorance of the unpolarized initial spin configurations. Then, integrating over the azimuth angle ϕ, we arrive at

$$\boxed{\left(\frac{d\sigma}{d\cos\theta}\right)_{\text{CM}} = \frac{e^4 m_e^4}{16\pi E_{\text{CM}}^2 (E_{\text{CM}}^2 - 4m_e^2)^2} \frac{1 + 3\cos^2\theta}{\sin^4\theta}}. \tag{7.19}$$

The angular distribution is shown in Fig. 7.2.

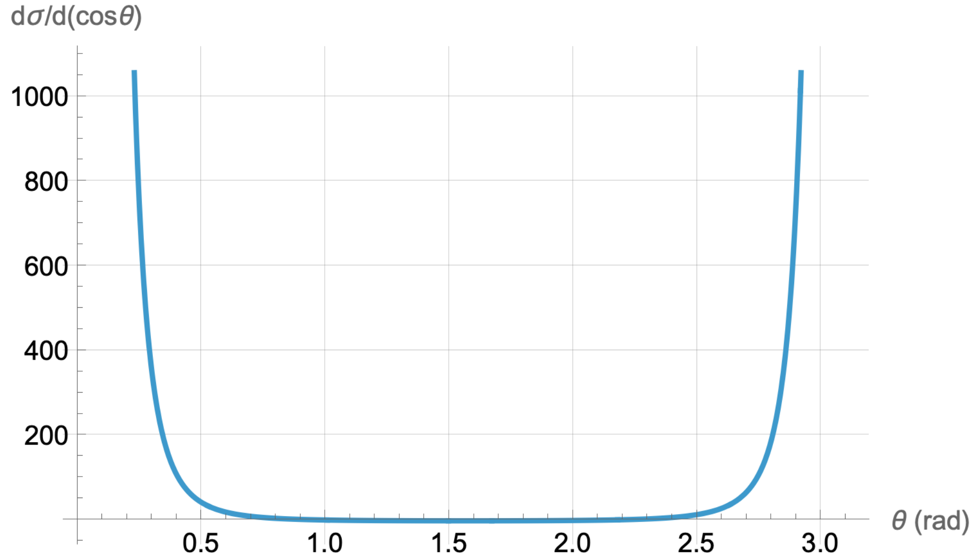
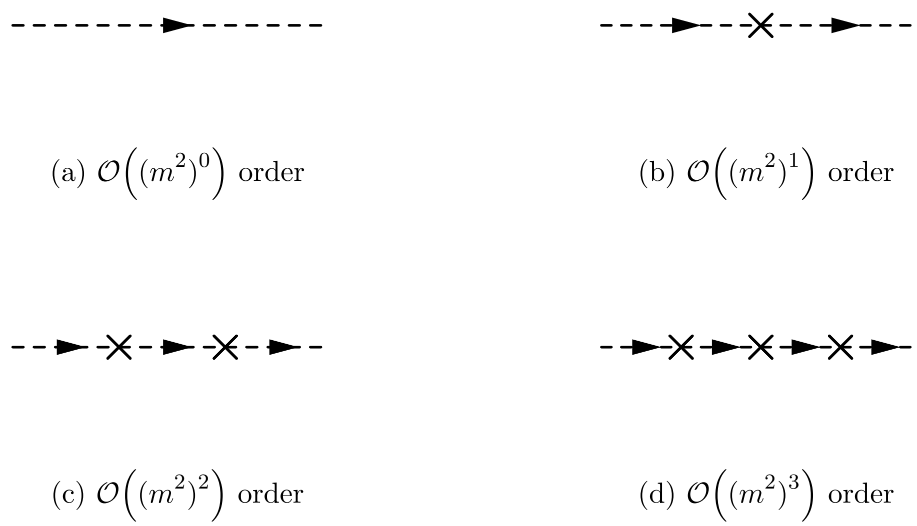


Fig. 7.2: The angular distribution

7.4

(a) The Feynman diagrams for the 2-point function are shown in Fig. 7.3. The cross denotes a "mass vertex" insertion.



(a) $\mathcal{O}\Big((m^2)^0\Big)$ order

(b) $\mathcal{O}\Big((m^2)^1\Big)$ order

(c) $\mathcal{O}\Big((m^2)^2\Big)$ order

(d) $\mathcal{O}\Big((m^2)^3\Big)$ order

Fig. 7.3: Mass insertions contributing to the 2-point function.

(b) The momentum-space Green's function $G(p^2)$ is formally defined via the 2-point function:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} iG(p^2). \tag{7.20}$$

Let $G_n(p^2)$ denote the $\mathcal{O}\big((m^2)^n\big)$ contribution. At tree level, this is just the massless propagator,

$$iG_0 \equiv \frac{i}{p^2 + i\varepsilon}. \tag{7.21}$$

At higher orders,

$$iG_1 = iG_0(-im^2)iG_0, \tag{7.22}$$
$$iG_2 = iG_0(-im^2)iG_0(-im^2)iG_0, \tag{7.23}$$
$$iG_3 = iG_0(-im^2)iG_0(-im^2)iG_0(-im^2)iG_0. \tag{7.24}$$

(c)

$$\begin{aligned}
iG(p^2) &= iG_1 + iG_2 + iG_3 + \cdots \\
&= \frac{i}{p^2+i\varepsilon} + \frac{i}{p^2+i\varepsilon}(-im^2)\frac{i}{p^2+i\varepsilon} + \frac{i}{p^2+i\varepsilon}(-im^2)\frac{i}{p^2+i\varepsilon}(-im^2)\frac{i}{p^2+i\varepsilon} + \cdots \\
&= \frac{i}{p^2+i\varepsilon}\sum_{n=0}^{\infty}\left(\frac{m^2}{p^2+i\varepsilon}\right)^n \\
&= \frac{i}{p^2+i\varepsilon}\frac{1}{1-\frac{m^2}{p^2+i\varepsilon}} \\
&= \frac{i}{p^2-m^2+i\varepsilon},
\end{aligned} \tag{7.25}$$

where we treat the "mass interaction" to be small ($m^2 \ll p^2$) in the sense that the perturbation series does not break down, such that the geometric series is well-defined. The final expression reproduces the propagator that one would have obtained directly for a massive scalar field.

(d) We can set up the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2 + J\phi. \tag{7.26}$$

The equation of motion are

$$\Box\phi = J - m^2\phi. \tag{7.27}$$

Solving this perturbatively with $\phi_n = \mathcal{O}\big((m^2)^n\big)$, we have

- $\boldsymbol{\mathcal{O}\big((m^2)^0\big)}$**:**

$$\Box\phi_0 = J + \mathcal{O}\big((m^2)^1\big). \tag{7.28}$$

- $\boldsymbol{\mathcal{O}\big((m^2)^1\big)}$**:**

$$\Box(\phi_0+\phi_1) = J - m^2\phi_0 + \mathcal{O}\big((m^2)^2\big) = J\left(1 - \frac{m^2}{\Box}\right) + \mathcal{O}\big((m^2)^2\big). \tag{7.29}$$

- $\mathcal{O}((m^2)^2)$:

$$\Box(\phi_0+\phi_1+\phi_2) = J - m^2(\phi_0+\phi_1) + \mathcal{O}((m^2)^3) = J\left(1-\frac{m^2}{\Box}\left(1-\frac{m^2}{\Box}\right)\right) + \mathcal{O}((m^2)^3). \tag{7.30}$$

Continuing the perturbation series, one can deduce that

$$\Box\phi = \Box\left(\sum_{n=0}^{\infty}\phi_n\right) = J\sum_{n=0}^{\infty}\left(-\frac{m^2}{\Box}\right)^n = J\frac{1}{1+\frac{m^2}{\Box}}, \tag{7.31}$$

and hence

$$\phi = J\frac{1}{\Box+m^2}. \tag{7.32}$$

However, this solves exactly the same equation of motion if one included the mass to begin with:

$$(\Box+m^2)\phi = J. \tag{7.33}$$

7.5

Besides the proof provided in Subsection 7.4.2 of the textbook, I present here an alternative demonstration that integrating by parts does not affect matrix elements. The argument once again reduces to showing that a total derivative term does not contribute in *perturbation theory*. Consider modifying the Lagrangian by adding a total derivative:

$$\mathcal{L} \to \mathcal{L} + \partial_\mu X_\mu. \tag{7.34}$$

Then the action shifts as

$$S \to \int d^4x(\mathcal{L}+\partial_\mu X_\mu) = S + \oint_{S^3_\infty} X_\mu d\Sigma_\mu. \tag{7.35}$$

If we assume that X^μ is constructed from operators that asymptotically vanish at spatial and temporal infinity—i.e., they **approach the same trivial vacuum**, ensuring $X^\mu \to 0$ at spacetime infinity—then the surface integral vanishes, and thus $\delta S \to 0$. Since the action remains unchanged, the matrix elements consequently remain unchanged as well.

However, the assumption in **boldface** is valid only in *perturbation theory*. In *non-perturbative theory*, classical vacua that are topologically distinct from the trivial vacuum may exist, causing the surface term to remain. In such cases, the above argument no longer applies.

7.6

There are four Feynman diagrams contributing to the $\phi_1\phi_2 \to \phi_1\phi_2$ scattering process, shown in Fig. 7.4. Two correspond to s-channel and two to u-channel diagrams. In each channel, the intermediate propagator can be either a ϕ_1 or a ϕ_2.

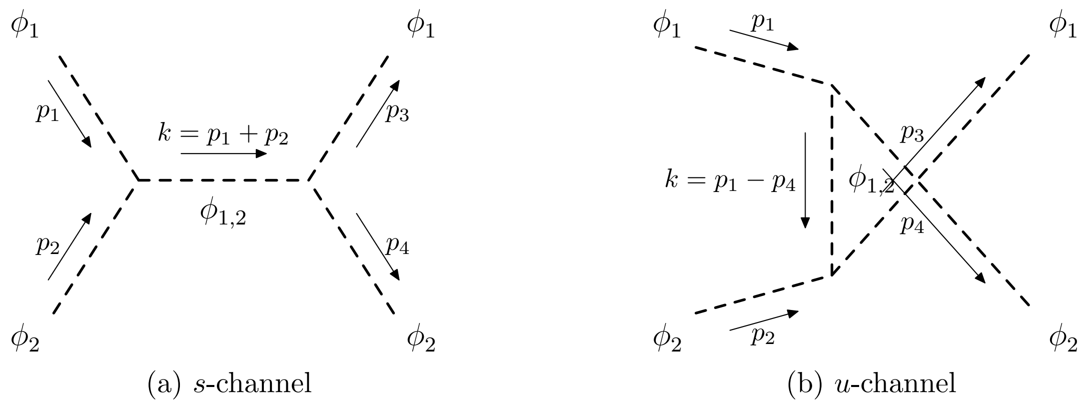


Fig. 7.4: Feynman diagrams for $\phi_1\phi_2 \to \phi_1\phi_2$ scattering.

From the kinetic terms in the Lagrangian, we know both ϕ_1 and ϕ_2 are massless:

$$p^2_{\phi_1,\phi_2} = 0. \tag{7.36}$$

Also,

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_1 \cdot p_2 = 2p_3 \cdot p_4, \tag{7.37}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -2p_1 \cdot p_4 = -2p_2 \cdot p_3, \tag{7.38}$$

$$s + t + u = 0. \tag{7.39}$$

Let us denote $\mathcal{M}_{s,i}$ as the s-channel amplitude mediated by ϕ_i, and similarly for $\mathcal{M}_{u,i}$. Then,

$$i\mathcal{M}_{s,1} = (ig)\frac{i}{k^2}(ig) = -i\frac{g^2}{s}, \tag{7.40}$$

$$i\mathcal{M}_{s,2} = (i\lambda)(-ip_2^\mu)(ik^\mu)\frac{i}{k^2}(i\lambda)(ip_4^\nu)(-ik^\nu) = -i\frac{\lambda^2}{s}(p_1 \cdot p_2 + p_2^2)(p_3 \cdot p_4 + p_4^2) = -i\frac{\lambda^2 s}{4}, \tag{7.41}$$

$$i\mathcal{M}_{u,1} = (ig)\frac{i}{k^2}(ig) = -i\frac{g^2}{u}, \tag{7.42}$$

$$i\mathcal{M}_{u,2} = (i\lambda)(ip_4^\mu)(ik^\mu)\frac{i}{k^2}(i\lambda)(-ip_2^\nu)(-ik^\nu) = -i\frac{\lambda^2}{t}(p_1^2 - p_1 \cdot p_4)(p_2^2 - p_2 \cdot p_3) = -i\frac{\lambda^2 u}{4}. \tag{7.43}$$

Combining all contributions,

$$i\mathcal{M} = i\mathcal{M}_{s,1} + i\mathcal{M}_{s,2} + i\mathcal{M}_{u,1} + i\mathcal{M}_{u,2} = -ig^2\left(\frac{1}{s} + \frac{1}{u}\right) - i\frac{\lambda^2}{4}(s+u) = i\frac{g^2 t}{su} + i\frac{\lambda^2 t}{4}, \tag{7.44}$$

and thus,

$$|\mathcal{M}|^2 = \left(\frac{g^2 t}{su} + \frac{\lambda^2 t}{4}\right)^2. \tag{7.45}$$

Plugging into Eq. (5.33) of the textbook,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{64\pi^2 E^2_{\text{CM}}}|\mathcal{M}|^2 = \boxed{\frac{1}{64\pi^2 s}\left(\frac{g^2 t}{su} + \frac{\lambda^2 t}{4}\right)^2}. \tag{7.46}$$

7.7

(a) The diagram contains 6 internal lines and 4 vertices. Let the incoming momenta be $p_{1,2}$ and the outgoing momenta be $p_{3,4}$. Label the internal momenta as follows:

- At the vertex linked with p_1: k_1, k_2, k_3.
- At the vertex linked with p_2: k_3, k_4, k_5.
- At the vertex linked with p_3: k_2, k_5, k_6.
- At the vertex linked with p_4: k_1, k_4, k_6.

The momentum conservation at the four vertex gives the following equations:

$$k_1 + k_2 - k_3 = p_1, \tag{7.47}$$
$$k_2 - k_5 + k_6 = p_3, \tag{7.48}$$
$$k_3 + k_4 - k_5 = p_2, \tag{7.49}$$
$$k_1 + k_4 - k_6 = p_4. \tag{7.50}$$

However, only 3 of these 4 equations are linearly independent. To see this, write them in matrix form $\mathbf{A} \cdot \mathbf{K} = \mathbf{P}$, with

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \tag{7.51}$$

where clearly, $\text{rank}(\mathbf{A}) = 3$.

We may choose k_2, k_3, k_4 as the independent loop momenta. Then,

$$k_1 = p_1 - k_2 + k_3, \tag{7.52}$$
$$k_5 = -p_2 + k_3 + k_4, \tag{7.53}$$
$$k_6 = p_3 - p_2 - k_2 + k_3 + k_4. \tag{7.54}$$

The amplitude is then given by

$$\begin{aligned} i\mathcal{M} &= (-i\lambda)^4 \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \prod_{i=1-6} \frac{i}{k_i^2 + i\varepsilon} \\ &= -\lambda^4 \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \left[\frac{i}{(p_1 - k_2 + k_3)^2 + i\varepsilon} \frac{i}{k_2^2 + i\varepsilon} \frac{i}{k_3^2 + i\varepsilon} \right. \\ &\quad \left. \times \frac{i}{k_4^2 + i\varepsilon} \frac{i}{(-p_2 + k_3 + k_4)^2 + i\varepsilon} \frac{i}{(p_3 - p_2 - k_2 + k_3 + k_4)^2 + i\varepsilon} \right]. \end{aligned} \tag{7.55}$$

Note that there is no non-trivial way to map this diagram onto itself without tangling up the external legs. Thus, the diagram has no non-trivial symmetry, and the symmetry factor is $\boxed{1}$.

(b) Chopping off the external lines, the symmetry of the diagram is that of a tetrahedron, which is isomorphic to the symmetric group S_4 with 24 elements. Therefore, the symmetry factor is $\boxed{24}$. This includes 12 proper rotations, 6 pure reflections, and 6 rotatory reflections that map the tetrahedron onto itself.

Alternatively, one can interpret the symmetry as arising from all permutations of the 4 vertices of the diagram, yielding $4! = 24$ distinct configurations. After all, S_4 is precisely the permutation group of four objects.

7.8

(a)

$$\boxed{\begin{aligned}\mathcal{L} = -\frac{1}{2}\phi_W(\Box + m_W^2)\phi_W + |\partial_\mu\phi_\mu|^2 - m_\mu^2|\phi_\mu|^2 + |\partial_e\phi_e|^2 + |\partial_\mu\phi_{\nu_\mu}|^2 + |\partial_\mu\phi_{\nu_e}|^2 \\ + g\phi_\mu\phi_W\phi^*_{\nu_\mu} + g\phi_e\phi_W\phi^*_{\nu_e}\end{aligned}}, \tag{7.56}$$

where $\phi_{\mu,e,\nu_\mu,\nu_e}$ must be complex scalars for them to have corresponding antiparticle. For these fields, we define $\phi_{\bar{x}} \equiv \phi_x^*$.

(b)

$$i\mathcal{M} = (ig)\frac{i}{k^2 - m_W^2}(ig), \tag{7.57}$$

where $k = p_\mu - p_{\nu_\mu}$ is the four-momentum of the intermediate W boson. Then,

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{g^4}{(k^2 - m_W^2)^2} \\ &= \frac{g^4}{m_W^4}\frac{1}{\left(\frac{k^2}{m_W^2} - 1\right)^2} \\ &\approx \boxed{\frac{g^4}{m_W^4}}, \qquad \frac{k^2}{m_W^2} \ll 1.\end{aligned} \tag{7.58}$$

In the rest frame of the muon:

$$p_\mu = (m_\mu, 0), \tag{7.59}$$

$$p_{\nu_\mu} = (E_{\nu_\mu}, \vec{p}_{\nu_\mu}), \tag{7.60}$$

where $E_{\nu_\mu} = |\vec{p}_{\nu_\mu}|$ since we treat the neutrino as massless.[1] We then observe

$$k^2 = (p_\mu - p_{\nu_\mu})^2 = m_\mu(m_\mu - 2E_{\nu_\mu}). \tag{7.61}$$

Note that $E_{\nu_\mu} < m_\mu/2$, otherwise energy and momentum conservation would be violated. Therefore, the $\frac{k^2}{m_W^2} \ll 1$ holds as long as $\frac{m_\mu^2}{m_W^2} \ll 1$.

[1] Although the problem in part (a) assumes $m_\mu = 0$, a massless particle cannot decay, so we must keep m_μ here.

(c) The dimension of the decay rate Γ is

$$[\Gamma] = 1. \tag{7.62}$$

To ensure that $[g] = 0$, we must insert a factor of m_μ^5 to restore the correct mass dimension of the decay rate. Hence,

$$\boxed{\Gamma = \frac{g^4}{192\pi^3}\frac{m_\mu^5}{m_W^4}}\,. \tag{7.63}$$

(d) Suppose $g \sim e = 0.303$. Given that $\Gamma = \frac{1}{2.2\times 10^{-6}\ \text{s}}$ and $m_\mu = 0.105$ GeV, we have

$$m_W = g\left(\frac{1}{192\pi^3}\frac{m_\mu^5}{\hbar\Gamma}\right)^{1/4} = \boxed{88.2\ \text{GeV}}\,, \tag{7.64}$$

where I have inserted an $\hbar$ to convert the decay rate into energy units.

(e) By assumption, the coupling strengths are the same, and the τ decay is also mediated by the W boson. Hence,

$$\frac{\Gamma_\tau}{\Gamma_\mu} = \left(\frac{m_\tau}{m_\mu}\right)^5$$

$$m_\tau = \left(\frac{\Gamma_\tau}{\Gamma_\mu}\right)^{1/5} m_\mu \approx \boxed{2.50\ \text{GeV}}\,, \tag{7.65}$$

where I have plugged in $\Gamma_\tau^{-1} = 2.9\times 10^{-13}$ s.

(f) As

$$\frac{\Gamma(\tau\to e^-\nu_e\nu_\tau)}{\Gamma_{\text{tot}}} = 0.178, \tag{7.66}$$

we have

$$m_\tau = \left(\frac{0.178\times\Gamma_{\text{tot}}}{\Gamma_\mu}\right)^{1/5} m_\mu \approx \boxed{1.77\ \text{GeV}}\,. \tag{7.67}$$

(g) As "assumed-to-be" scalars, their decay distributions are isotropic. Moreover, since all the tree-level decay rates Γ_{tree} depend only on the combination $\frac{g}{m_W}$ up to some power, it is not possible to extract g and m_W separately at tree level.

Since e and μ are electrically charged, one approach is to consider NLO corrections involving photon vertex correction. An example is shown in Eq. (23.38) of the textbook. The decay width, including the NLO correction from a photon[2], takes the form

$$\Gamma_\mu \sim \Gamma_{\mu,\text{tree}}\left[1 + \frac{\alpha}{4\pi}F_\mu\left(\frac{m_W}{m_\mu}\right) + \cdots\right], \tag{7.68}$$

[2]Since m_W is very heavy and $g \sim e$ by assumption, the leading NLO correction arises from photon vertex contributions rather than W-mediated corrections.

where F_μ is a dimensionless form factor. While its precise functional form does not matter here, it must depend only on the dimensionless ratio $\frac{m_W}{m_\mu}$ by dimensional analysis, since m_W and m_μ are the only relevant energy scales (electron and neutrino masses are neglected).

As the correction term depends solely on m_W and not on g, it is in principle possible to extract m_W by comparing the measured decay rate to the tree-level prediction. However, note that one cannot extract m_W by observing only muon decay. The reason is that we don't really know the values of $\frac{g}{m_W}$. What we actually measure is the total decay rate, which includes both the tree-level contribution and all radiative corrections. Hence, we can't properly measure even the ratio itself.

The strategy is to compare the deviation in the muon decay with that in the tau decay, and to note that the tree-level ratio satisfies $\frac{\Gamma_{\tau,\text{tree}}}{\Gamma_{\mu,\text{tree}}} = \left(\frac{m_\tau}{m_\mu}\right)^5$. We can see

$$\frac{\Gamma_\tau}{\Gamma_\mu} \sim \left(\frac{m_\tau}{m_\mu}\right)^5 \frac{1 + \frac{\alpha}{4\pi} F_\mu \left(\frac{m_W}{m_\tau}\right) + \cdots}{1 + \frac{\alpha}{4\pi} F_\tau \left(\frac{m_W}{m_\mu}\right) + \cdots}, \tag{7.69}$$

of which m_W dependence is isolated.

Once a value for m_W is extracted from this, it can be substituted back into Eq. (7.68) to determine g.

Hence, to see the difference in the NLO corrections, one needs to measure both tau and muon decays with precision better than $\mathcal{O}\left(\frac{\alpha}{4\pi} F\right)$.

7.9

(a) The tree-level cross section is proportional to

$$\sigma \propto \frac{1}{s} |\mathcal{M}|^2 \propto \frac{1}{s} \left| \frac{i}{s - m^2 + im\Gamma} \right|^2 = \boxed{\frac{1}{s} \frac{1}{(s - m^2)^2 + m^2\Gamma^2}}. \tag{7.70}$$

(b) The sketch is shown in Fig. 7.5.

(c) Technically, this should not be labeled as the amplitude $\mathcal{M}$, whose mass dimension is $[\mathcal{M}] = 0$, which is not the case for a propagator.

$$\begin{aligned} \operatorname{Im} \frac{1}{p^2 - m^2 + i\varepsilon} &= \frac{1}{2i} \left(\frac{1}{p^2 - m^2 + i\varepsilon} - \frac{1}{p^2 - m^2 - i\varepsilon} \right) \\ &= \frac{-\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2}, \end{aligned} \tag{7.71}$$

which vanishes as $\varepsilon \to 0$ unless evaluated on-shell, $p^2 = m^2$.

Note if one integrates over p^2,

$$\int_0^\infty dp^2 \frac{-\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2} = -\left[\tan^{-1}\left(\frac{p^2 - m^2}{\varepsilon}\right)\right]_0^\infty \xrightarrow{\varepsilon \to 0} -\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi. \tag{7.72}$$

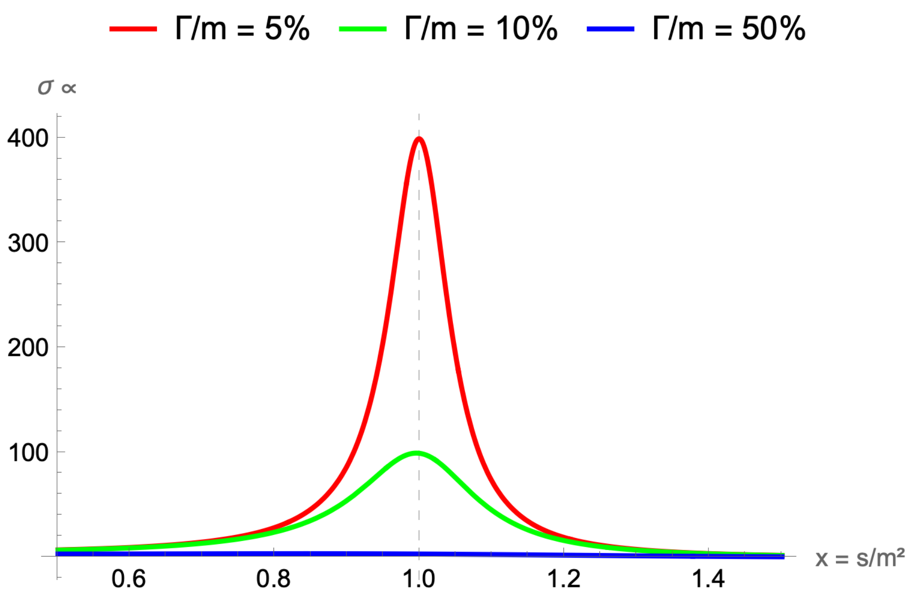


Fig. 7.5

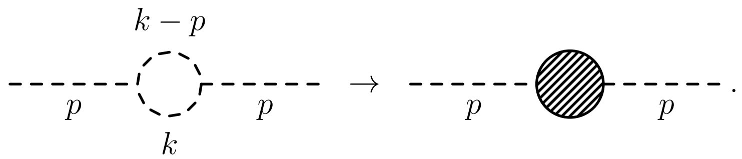
Since one also has

$$\int_0^\infty dp^2(-\pi)\delta(p^2 - m^2) = -\pi, \tag{7.73}$$

it follows that

$$\boxed{\text{Im}\,\frac{1}{p^2 - m^2 + i\varepsilon} = -\pi\delta(p^2 - m^2)}\,. \tag{7.74}$$

(d) Suppose the interaction term is $\frac{g}{2}\phi\psi\psi$. For simplicity, we treat ψ as a scalar. Let the mass of ϕ be M and the mass of ψ be m. We can describe the loop diagram (excluding the initial and final lines) using an "effective" interaction:



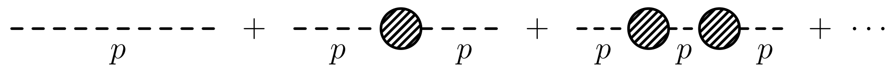
(7.75)

That is

$$i\mathcal{M}_{\text{loop}}(p) = \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} \to i\Sigma(p). \tag{7.76}$$

We shall also denote the Feynman propagator as $\Pi_F(p) \equiv \frac{i}{p^2 - M^2 + i\varepsilon}$. Then, the dressed propagator $iG(p)$ can be written as a perturbative sum over the "effective" interaction at all orders:

$iG(p) =$



$$
\begin{aligned}
&= \Pi_F(p) + \Pi_F(p)(i\Sigma(p))\Pi_F(p) + \Pi_F(p)(i\Sigma(p))\Pi_F(p)(i\Sigma(p))\Pi_F(p) + \cdots \\
&= \Pi_F(p)\sum_{n=0}^{\infty}(i\Sigma(p)\Pi_F(p))^n \\
&= \Pi_F(p)\frac{1}{1 - i\Sigma(p)\Pi_F(p)} \\
&= \frac{i}{p^2 - M^2 + \Sigma(p) + i\varepsilon}.
\end{aligned}
\tag{7.77}
$$

The Breit-Wigner distribution is then reproduced if one redefines $M^2 \to M^2 + \mathrm{Re}\,\Sigma(p)$ (as a spoiler, this is actually the mass renormalization), and identifies $\mathrm{Im}\,\Sigma(p) = M\Gamma$.

(e) The result of part (d) says

$$
\mathrm{Im}\,\Sigma(p) = \mathrm{Im}\,\mathcal{M}_{\text{loop}} = M\Gamma_{\text{BW}}. \tag{7.78}
$$

What we need to prove is that the width Γ_{BW} appearing in the Breit-Wigner distribution is exactly the decay rate $\Gamma_{\phi\to\psi\psi}$. This is actually one of the implications of the optical theorem. The steps below follow Section 24.1 of the textbook.

From Eq. (6.19), we also have[3]

$$
\Pi_F(k) = \Pi_A(k) + \frac{\pi}{\omega_k}\delta(k_0 - \omega_k), \tag{7.79}
$$

where the advanced propagator is given by

$$
\Pi_A(k) = \frac{i}{2\omega_k}\left[\frac{1}{k_0 - (\omega_k + i\varepsilon)} - \frac{1}{k_0 - (-\omega_k + i\varepsilon)}\right]. \tag{7.80}
$$

The derivation of part (c) can also be used to show

$$
\mathrm{Im}\,\frac{1}{k_0 - \omega_k + i\varepsilon} = -\pi\delta(k_0 - \omega_k). \tag{7.81}
$$

Plugging this back into the loop expression in Eq. (7.76):

$$
i\mathcal{M}_{\text{loop}}(p) = -\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\left[\Pi_A(k-p) + \frac{\pi}{\omega_{k-p}}\delta(k_0 - p_0 - \omega_{k-p})\right]\left[\Pi_A(k) + \frac{\pi}{\omega_k}\delta(k_0 - \omega_k)\right]. \tag{7.82}
$$

The product term $\Pi_A(k-p)\Pi_A(k)$ can be dropped because both propagators have poles located above the real k_0 axis. Thus, when performing the k_0 integral, one can safely close

[3] We swapped "retarded" and "advanced" here in accordance with the textbook convention.

the contour in the lower half-plane without enclosing any poles, resulting in a vanishing contribution.

Similarly, the product of the two δ-functions can be dropped because their arguments cannot simultaneously vanish. For instance, in the rest frame where $\vec{p} = 0$ so that $p_0 = M$ and $\omega_{k-p} = \omega_k$, the two delta conditions $\delta(k_0 - \omega_k)$ and $\delta(k_0 - p_0 - \omega_{k-p})$ are mutually exclusive. Therefore, their product has no support.

Thus,

$$\begin{aligned}\mathcal{M}_{\text{loop}}(p) &= \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\left[i\Pi_A(k-p)\frac{\pi}{\omega_k}\delta(k_0-\omega_k) + \frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})i\Pi_A(k)\right] \\ &= \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\left[i\Pi_F(k-p)\frac{\pi}{\omega_k}\delta(k_0-\omega_k) + \frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})i\Pi_F(k)\right],\end{aligned} \tag{7.83}$$

where we used Eq. (7.79) again, and dropped the product of two delta functions to get the last line.

Taking the imaginary part. Note that the delta functions are real, so the imaginary part only comes from $i\Pi_F$. Using Eq. (7.74), we find

$$\operatorname{Im}\mathcal{M}_{\text{loop}}(p) = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\left[\pi\delta\left((k-p)^2-m^2\right)\frac{\pi}{\omega_k}\delta(k_0-\omega_k) + \pi\delta(k^2-m^2)\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\right]. \tag{7.84}$$

The second term vanishes since the delta functions cannot be simultaneously satisfied. Now we use the identity

$$\frac{1}{2\omega_k}\delta(k_0-\omega_k) = \delta(k^2-m^2) - \frac{1}{2\omega_k}\delta(k_0+\omega_k), \tag{7.85}$$

and again drop the term $\delta\left((k-p)^2-m^2\right)\delta(k_0+\omega_k)$. We obtain

$$\operatorname{Im}\mathcal{M}_{\text{loop}}(p) = -\frac{g^2}{4}\int \frac{d^4k}{(2\pi)^4}(-2\pi i)\delta\left((k-p)^2-m^2\right)(-2\pi i)\delta(k_0-\omega_k). \tag{7.86}$$

We just derived cutting rules.

Lastly, we change variables by letting $k = q_2$ and $k - p = q_1$, and insert the identity $1 = \int d^4q_1\delta^4(p - q_1 - q_2)$:

$$\operatorname{Im}\mathcal{M}_{\text{loop}}(p) = \frac{g^2}{4}\int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}(2\pi)^6\delta(q_1^2-m^2)\delta(q_2^2-m^2)\delta^4(p-q_1-q_2). \tag{7.87}$$

Recall from Eq. (2.36), as shown in Problem 2.6,

$$\int \frac{d^4q}{(2\pi)^4}2\pi\delta(q^2-m^2)\theta(q_0) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q} \tag{7.88}$$

and note that since $p^0 > 0$, the δ-functions can only have support for $q_1^0 > 0$ and $q_2^0 > 0$, we then have

$$\begin{aligned}\operatorname{Im}\mathcal{M}_{\text{loop}}(p) &= \frac{g^2}{4}(2\pi)^4\delta^4(p-q_1-q_2)\int\frac{d^3q_1}{(2\pi)^3}\int\frac{d^3q_2}{(2\pi)^3}\frac{1}{2\omega_{q_1}}\frac{1}{2\omega_{q_2}}\\ &= \frac{g^2}{4}\int d\Pi_{\text{LIPS}},\end{aligned} \tag{7.89}$$

where the integral is nothing but the two-body Lorentz-invariant phase space.

On the other hand, the tree-level decay amplitude for $\phi \to \psi\psi$ is similar to that in Eq. (7.1). Using Eq. (5.24) of the textbook, in the rest frame of ϕ, we have

$$\Gamma_{\phi\to\psi\psi} = \frac{1}{2}\frac{g^2}{2M}\int d\Pi_{\text{LIPS}}, \tag{7.90}$$

where the factor of $\frac{1}{2}$ accounts for the two identical final-state particles ψ.

Comparing Eq. (7.89) with Eq. (7.90), we find

$$\boxed{M\Gamma_{\text{BW}} = \operatorname{Im}\Sigma = \operatorname{Im}\mathcal{M}_{\text{loop}} = M\Gamma_{\phi\to\psi\psi}}\,. \tag{7.91}$$

Therefore, the width in the Breit-Wigner resonance distribution is exactly the decay rate.

Part II

Quantum electrodynamics

Chapter 8

Spin 1 and gauge invariance

8.1

Consider two arbitrary states $|\psi\rangle$ and $\left|\psi'\right\rangle$ in Hilbert space. We are interested in determining the probability of projecting $\left|\psi'\right\rangle$ onto $|\psi\rangle$. Suppose the unnormalized $P \equiv |\left\langle\psi\middle|\psi'\right\rangle|^2 > 1$. To retain a sensible probabilistic interpretation (i.e., $P \leq 1$), we can normalize this projection by the norms of each state. This is valid, as states in Hilbert space are *rays* rather than *vectors*. Thus, we have

$$P = \frac{|\left\langle\psi\middle|\psi'\right\rangle|^2}{\langle\psi|\psi\rangle\left\langle\psi'\middle|\psi'\right\rangle} \leq 1. \tag{8.1}$$

This, however, implies[1]

$$|\left\langle\psi\middle|\psi'\right\rangle|^2 \leq \langle\psi|\psi\rangle\left\langle\psi'\middle|\psi'\right\rangle. \tag{8.2}$$

The left-hand side $|\left\langle\psi\middle|\psi'\right\rangle|^2$ is manifestly non-negative. If the two norms on the right-hand side had opposite signs, this inequality would be violated. Therefore, consistency requires that both $\langle\psi|\psi\rangle$ and $\left\langle\psi'\middle|\psi'\right\rangle$ be either positive (positive-definite norm) or both negative (negative-definite norm).

8.2

$$\begin{aligned}\mathcal{T}_{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\partial_\nu A_\alpha - g_{\mu\nu}\mathcal{L} \\ &= \boxed{-F_{\mu\alpha}\partial_\nu A_\alpha + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}^2}\,.\end{aligned} \tag{8.3}$$

The energy density ε is

$$\begin{aligned}\varepsilon = \mathcal{T}_{00} &= -F_{0\alpha}\partial_t A_\alpha + \frac{1}{4}F_{\alpha\beta}^2 \\ &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + A_0\partial_t(\partial_\mu A_\mu) - A_0\Box A_0 + \partial_i(A_0F_{0i}),\end{aligned} \tag{8.4}$$

[1] One might recognize this is simply the *Cauchy–Schwarz inequality*. However, we refrain from quoting it outright. This is one of the place where physicists and mathematicians diverge from each others because we don't assume the norm $\left\langle\psi\middle|\psi'\right\rangle$ to be positive-definite, while a mathematician would argue a "*norm*" is, *by definition*, positive-definite, and this problem is mathematically *trivial*.

which easily follows by taking the $m \to 0$ limit of Eq. (8.27) in the textbook. The first term is positive-definite. The second term vanishes under the Lorenz gauge $\partial_\mu A_\mu = 0$. The third term drops out once the equation of motion $\Box A_0 = 0$ is imposed. Thus, the only possibly non-positive contribution arises from the last term $\partial_i(A_0 F_{0i})$, which is a total spatial divergence. Therefore, we conclude that

$$\boxed{\varepsilon - \partial_i(A_0 F_{0i}) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) > 0}\,, \tag{8.5}$$

and we recognize that $\boxed{X_i = A_0 F_{0i}}$.

8.3

The classical Lagrangian for a massive spin-1 particle (i.e., the *Proca Lagrangian*) with a source current J_μ is given by

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \tag{8.6}$$

Applying the Euler–Lagrange equation yields the equation of motion

$$\Box A_\mu - \partial_\mu \partial_\nu A_\nu + m^2 A_\mu = J_\mu, \tag{8.7}$$

or in momentum space,

$$\left[(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\right] A_\nu = J_\mu. \tag{8.8}$$

By Lorentz invariance, the propagator must take the general form

$$\Pi_{\mu\nu} = A g_{\mu\nu} + B p_\mu p_\nu, \tag{8.9}$$

where A and B are scalar functions that may depend on p^2 and m^2. Substituting the inversion $A_\mu = \Pi_{\mu\nu} J_\nu$ back into Eq. (8.8), we have

$$\begin{aligned}
\left[(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\right] A_\nu &= J_\mu \\
\left[(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\right] \Pi_{\nu\alpha} J_\alpha &= J_\mu \\
\left[(-p^2 + m^2)g_{\mu\nu} + p_\mu p_\nu\right] (A g_{\nu\alpha} + B p_\nu p_\alpha) J_\alpha &= J_\mu \\
[A((-p^2 + m^2)g_{\mu\alpha} + p_\mu p_\alpha) + B((-p^2 + m^2)p_\mu p_\alpha + p^2 p_\mu p_\alpha)] J_\alpha &= J_\mu \\
A(-p^2 + m^2)g_{\mu\alpha} + (A + Bm^2)p_\mu p_\alpha &= g_{\mu\alpha}.
\end{aligned} \tag{8.10}$$

Matching coefficients, we have

$$A(-p^2 + m^2) = 1, \tag{8.11}$$

$$A + Bm^2 = 0. \tag{8.12}$$

Solving this system yields

$$A = \frac{-1}{p^2 - m^2} \tag{8.13}$$

$$B = \frac{\frac{1}{m^2}}{p^2 - m^2}. \tag{8.14}$$

Thus, the classical propagator for a massive spin-1 particle takes the form

$$\boxed{\Pi_{\mu\nu} = A g_{\mu\nu} + B p_\mu p_\nu = \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2}}\,. \tag{8.15}$$

8.4

I would suggest working out Problem 8.5 first before proceeding with this one.

To impose the axial gauge $A_0 = 0$, we can make use of the result in Eq. (8.45), and set the reference vector to be $r_\mu = (1, 0, 0, 0)$. This choice ensures $A_\mu \cdot r^\mu = 0$, which enforces $A_0 = 0$. The corresponding photon propagator then reads

$$\begin{aligned} \Pi_{00} &= \frac{1}{p^2}(-1 - 1 + 2) = 0, \\ \Pi_{0i} = \Pi_{i0} &= \frac{1}{p^2}\left(-\frac{p_i}{E} + \frac{p_i}{E}\right) = 0, \\ \Pi_{ij} = \Pi_{ji} &= \frac{1}{p^2}\left(\delta_{ij} - \frac{p_i p_j}{E^2}\right). \end{aligned} \tag{8.16}$$

To summarize, the photon propagator in axial gauge takes the form

$$\boxed{i\Pi_{ij} = \frac{i}{p^2}\left(\delta_{ij} - \frac{p_i p_j}{E^2}\right)}\,. \tag{8.17}$$

8.5

(a) Let us begin by considering the frame in which the spin-1 particle propagates along the z-axis, following Eq. (8.68) and Eq. (8.69) of the textbook. Its four-momentum is

$$p^\mu = (E, 0, 0, p_z), \tag{8.18}$$

and a suitable choice of polarization basis is

$$\epsilon^1_\mu(p) = (0, 1, 0, 0), \quad \epsilon^2_\mu(p) = (0, 0, 1, 0), \quad \epsilon^L_\mu(p) \equiv \epsilon^3_\mu(p) = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right). \tag{8.19}$$

Define a rank-2 tensor to represent the polarization sum, $P_{\mu\nu} \equiv \sum_i \epsilon^i_\mu \epsilon^i_\nu$[2]. In this basis, the only non-vanishing components are

$$\begin{aligned} P_{00} &= \sum_{i=1,2,3} \epsilon^i_0 \epsilon^i_0 = \frac{p_z^2}{m^2} = -1 + \frac{E^2}{m^2}, \\ P_{11} = P_{22} &= \sum_{i=1,2,3} \epsilon^i_1 \epsilon^i_1 = \sum_{i=1,2,3} \epsilon^i_2 \epsilon^i_2 = 1, \\ P_{33} &= \sum_{i=1,2,3} \epsilon^i_3 \epsilon^i_3 = \frac{E^2}{m^2} = 1 + \frac{p_z^2}{m^2}, \\ P_{03} = P_{30} &= \sum_{i=1,2,3} \epsilon^i_0 \epsilon^i_3 = \sum_{i=1,2,3} \epsilon^i_3 \epsilon^i_0 = \frac{E p_z}{m^2}. \end{aligned} \tag{8.20}$$

[2] Note that the outer product of two rank-1 tensors (i.e., vectors) yields a rank-2 tensor.

By Lorentz invariance, the polarization sum must take the general tensorial form

$$\sum_i \epsilon^i_\mu \epsilon^i_\nu \equiv P_{\mu\nu} = A g_{\mu\nu} + B p_\mu p_\nu, \tag{8.21}$$

where A and B are scalar functions that can depend on p^2 and m^2. Since the physical polarization vectors ϵ_μ are transverse to the four-momentum p_μ, we must have

$$0 = p_\mu \sum_i \epsilon^i_\mu \epsilon^i_\nu = p_\mu P_{\mu\nu} = (A + Bm^2) p_\nu \implies B = -A/m^2, \tag{8.22}$$

where the on-shell condition $p^2 = m^2$ has been used for the physical four-momentum.

Comparing the explicit expression in Eq. (8.20) with the general form in Eq. (8.21), we find the normalization $A = -1$. Therefore, the physical polarization sum is given by

$$\boxed{\sum_i \epsilon^i_\mu \epsilon^i_\nu \equiv P_{\mu\nu} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}\,. \tag{8.23}$$

(b) Compared with Eq. (8.15), it is evident that Eq. (8.23) precisely corresponds to its numerator. The propagator is formally defined via the correlation function. Recall the derivation in Chapter 6.2 of the textbook: for a free scalar field, one obtains

$$\langle 0| T\{\phi_0(x_1)\phi_0(x_2)\} |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip(x_1 - x_2)}. \tag{8.24}$$

For a massive spin-1 particle, following an analogous derivation and employing the field operator for a massive vector field (cf. Eq. (8.64) of the textbook), one arrives at a similar result, but with a polarization sum:

$$\begin{aligned} \langle 0| T\{A_\mu(x_1)A_\nu(x_2)\} |0\rangle &= i \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \Pi_{\mu\nu}(p) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i \sum_i \epsilon^i_\mu \epsilon^{i*}_\nu}{p^2 - m^2 + i\epsilon} e^{ip(x_1 - x_2)}. \end{aligned} \tag{8.25}$$

This explains why the numerator of the propagator corresponds to the polarization sum. Note there are no cross terms in the polarization sum, as the polarization vectors basis are orthogonal to each other by definition. Also, the summation should only include the physical polarizations.

(c) Start in the frame where the particle propagates along the z-axis, consistent with Eq. (8.77)–(8.78) of the textbook. In this frame, the four-momentum reads

$$p^\mu = (E, 0, 0, E), \tag{8.26}$$

and the polarization basis vectors are

$$\epsilon^\mu_1(p) = (0, 1, 0, 0),\ \epsilon^\mu_2(p) = (0, 0, 1, 0), \tag{8.27}$$

which result from choosing the reference vector

$$r_\mu = (E, 0, 0, -E). \tag{8.28}$$

These basis vectors clearly satisfy $\epsilon^i \cdot r = 0$. The only non-vanishing components of the polarization sum are

$$\sum_{i=1,2} \epsilon_1^i \epsilon_1^i = \sum_{i=1,2} \epsilon_2^i \epsilon_2^i = 1. \tag{8.29}$$

Now, by Lorentz invariance, the general structure of the polarization sum takes the form

$$\sum_{i=1,2} \epsilon_\mu^i \epsilon_\nu^i = A g_{\mu\nu} + B p_\mu p_\nu + C r_\mu r_\nu + D r_\mu p_\nu + E p_\mu r_\nu, \tag{8.30}$$

where A, B, C, D, E are scalar functions, potentially dependent on p^2 and m^2, that remain to be determined. From the condition $\epsilon^i \cdot p = 0$, we must have

$$0 = p_\mu \sum_{i=1,2} \epsilon_\mu^i \epsilon_\nu^i = A p_\nu + B p^2 p_\nu + C(r \cdot p) r_\nu + D(r \cdot p) p_\nu + E p^2 r_\nu. \tag{8.31}$$

Grouping the coefficients gives:

$$\begin{aligned} p_\nu &: A + Bp^2 + D(r \cdot p) = 0, \\ r_\nu &: C(r \cdot p) + Ep^2 = 0. \end{aligned} \tag{8.32}$$

Similarly, with $0 = p_\nu \sum_{i=1,2} \epsilon_\mu^i \epsilon_\nu^i$, we have

$$\begin{aligned} p_\mu &: A + Bp^2 + E(r \cdot p) = 0, \\ r_\mu &: C(r \cdot p) + Dp^2 = 0. \end{aligned} \tag{8.33}$$

From these, we deduce $D = E$. Since the particle is massless ($p^2 = 0$), it follows that $C = 0$. Using the condition $\epsilon^i \cdot r = 0$, we also find:

$$0 = r_\mu \sum_{i=1,2} \epsilon_\mu^i \epsilon_\nu^i = A r_\nu + B(r \cdot p) p_\nu + D r^2 p_\nu + D(r \cdot p) r_\nu. \tag{8.34}$$

Again, collecting terms:

$$\begin{aligned} p_\nu &: B(r \cdot p) + Dr^2 = 0, \\ r_\nu &: A + D(r \cdot p) = 0. \end{aligned} \tag{8.35}$$

Solving this linear system yields

$$B = \frac{Ar^2}{(r \cdot p)^2}, \tag{8.36}$$

$$C = 0, \tag{8.37}$$

$$D = E = -\frac{A}{(r \cdot p)}. \tag{8.38}$$

Substituting these back into Eq. (8.30), we now have

$$\sum_{i=1,2} \epsilon^i_\mu \epsilon^i_\nu = A\left[g_{\mu\nu} + \frac{r^2}{(r\cdot p)^2} p_\mu p_\nu - \frac{r_\mu p_\nu + p_\mu r_\nu}{r\cdot p}\right]. \tag{8.39}$$

Matching this against Eq. (8.29) determines the normalization $A = -1$. Therefore,

$$\boxed{\sum_{i=1,2} \epsilon^i_\mu \epsilon^i_\nu = -g_{\mu\nu} - \frac{r^2}{(r\cdot p)^2} p_\mu p_\nu + \frac{r_\mu p_\nu + p_\mu r_\nu}{r\cdot p}}. \tag{8.40}$$

(d) The condition $\epsilon^i \cdot r = 0$ is equivalent to $A_\mu \cdot r_\mu = 0$, so we can adopt an approach analogous to Eq. (8.98) of the textbook: introduce a gauge-fixing term $-\frac{1}{2\xi}(A_\mu \cdot r_\mu)^2$ into the Lagrangian, and then derive the equation of motion to verify that it yields the expected form of the propagator.

Since $\frac{\partial((A_\mu \cdot r_\mu)^2)}{\partial A_\nu} = 2(A_\mu \cdot r_\mu) r_\nu$, with the new term, the equations of motion for A_μ are

$$\left(-p^2 g_{\mu\nu} + p_\mu p_\nu - \frac{1}{\xi} r_\mu r_\nu\right) A_\nu = J_\mu. \tag{8.41}$$

To determine the propagator $\Pi_{\mu\nu}$, we seek an inverse tensor such that $A_\mu = \Pi_{\mu\nu} J_\nu$. By Lorentz invariance, the general ansatz for the propagator takes the form

$$\Pi_{\mu\nu} = A g_{\mu\nu} + B p_\mu p_\nu + C r_\mu r_\nu + D p_\mu r_\nu + E r_\mu p_\nu. \tag{8.42}$$

Symmetry under index exchange $\mu \leftrightarrow \nu$ requires $\boxed{E = D}$. Proceeding similarly as in Problem 8.3, we have

$$\begin{aligned} g_{\mu\nu} &= \left(-p^2 g_{\mu\alpha} + p_\mu p_\alpha - \frac{1}{\xi} r_\mu r_\alpha\right)(A g_{\alpha\nu} + B p_\alpha p_\nu + C r_\alpha r_\nu + D p_\alpha r_\nu + D r_\alpha p_\nu) \\ &= -A p^2 g_{\mu\nu} + A p_\mu p_\nu - A\frac{1}{\xi} r_\mu r_\nu - B\frac{1}{\xi} r_\mu (r\cdot p) p_\nu \\ &\quad - C p^2 r_\mu r_\nu + C p_\mu (p\cdot r) r_\nu - C\frac{1}{\xi} r^2 r_\mu r_\nu \\ &\qquad - D\frac{1}{\xi} r_\mu (p\cdot r) r_\nu - D p^2 r_\mu p_\nu + D p_\mu (p\cdot r) p_\nu - D\frac{1}{\xi} r_\mu r^2 p_\nu. \end{aligned} \tag{8.43}$$

Matching coefficients on both sides yields:

$$g_{\mu\nu} : -A p^2 = 1 \Rightarrow \boxed{A = -\frac{1}{p^2}}$$

$$p_\mu p_\nu : A + D(p\cdot r) = 0 \Rightarrow \boxed{D = -\frac{A}{p\cdot r} = \frac{1}{p^2}\frac{1}{p\cdot r}}$$

$$r_\mu r_\nu : -A\frac{1}{\xi} - C p^2 - C\frac{1}{\xi} r^2 - D\frac{1}{\xi}(p\cdot r) = 0 \Rightarrow C\left(\frac{1}{\xi} r^2 + p^2\right) = 0 \Rightarrow \boxed{C = 0}$$

$$r_\mu p_\nu : -B\frac{1}{\xi}(r\cdot p) - D p^2 - D\frac{1}{\xi} r^2 = 0 \Rightarrow \boxed{B = -D\left(p^2 + \frac{1}{\xi} r^2\right)\frac{\xi}{r\cdot p} = -\frac{1}{p^2}\frac{(\xi p^2 + r^2)}{(r\cdot p)^2} = -\frac{1}{p^2}\frac{r^2}{(r\cdot p)^2}}, \tag{8.44}$$

where in the final line I took the limit $\xi \to 0$ to enforce the condition $\epsilon^i \cdot r = 0$.

Thus, the photon propagator is given by

$$\boxed{i\Pi_{\mu\nu} = \frac{i}{p^2}\left[-g_{\mu\nu} - \frac{r^2}{(r\cdot p)^2}p_\mu p_\nu + \frac{r_\mu p_\nu + p_\mu r_\nu}{r\cdot p}\right]}, \tag{8.45}$$

whose numerator precisely matches Eq. (8.40).

(e) Unlike the photon propagator in the R_ξ gauges, this form of propagator contains only physical states. However, a trade-off is that gauge invariance is no longer manifest. Another drawback is that, once a reference vector is fixed, the propagator becomes non-covariant under Lorentz transformations.

Note in the case of QED, the Ward identity ensures that any expression contracted with the external momentum ultimately vanishes. As a result, physical observables remain gauge-independent, and calculations using this propagator yield the same results as those obtained in Feynman–'t Hooft gauge ($\xi = 1$): $i\Pi^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2+i\epsilon}$.

8.6

Warning: this problem is one of the occasion where the position of indices on a Lorentz-covariant object does matter.

(a) The symmetry condition for a 4-dimensional matrix eliminates $\frac{4(4-1)}{2} = 6$ degrees of freedom.

Then, the transversality conditions $k_\mu \epsilon^{(i)\ \mu\nu} = 0$ impose four independent constraints, removing another 4 degrees of freedom.

Altogether, these remove $\boxed{4 + 6 = 10}$ degrees of freedom.

(b) Choose the frame where the contravariant four-momentum is given by $k^\mu = (E, 0, 0, p_z)$. Its covariant form is then $k_\mu = g_{\mu\nu}k^\nu = (E, 0, 0, -p_z)$.

A general rank-2 tensor $\epsilon^{\mu\nu}$ can be represented by a matrix M, which admits a unique decomposition into symmetric and anti-symmetric components: $M_S = \frac{M+M^T}{2}$ and $M_A = \frac{M-M^T}{2}$. This decomposition respects Lorentz covariance, as Lorentz transformations preserve the (anti-)symmetry structure of tensors. Consequently, the symmetric and anti-symmetric parts correspond to distinct representations of the Lorentz group and transform independently.

The anti-symmetric component $\epsilon_A^{\mu\nu} \equiv M_A$ can be parameterized as

$$\epsilon_A^{\mu\nu} \equiv M_A = \begin{pmatrix} 0 & a_{01} & a_{02} & a_{03} \\ -a_{01} & 0 & a_{12} & a_{13} \\ -a_{02} & -a_{12} & 0 & a_{23} \\ -a_{03} & -a_{13} & -a_{23} & 0 \end{pmatrix}. \tag{8.46}$$

Imposing the transverse condition $k_\mu \epsilon_A^{(i)\ \mu\nu} = 0$, we obtain

$$\begin{aligned} \nu = 0 :&\ \ a_{03} = 0, \\ \nu = 1 :&\ \ Ea_{01} + p_z a_{13} = 0, \\ \nu = 2 :&\ \ Ea_{02} + p_z a_{23} = 0, \\ \nu = 3 :&\ \ a_{03} = 0. \end{aligned} \tag{8.47}$$

The general anti-symmetric matrix has 6 parameters, and 3 are fixed by the above constraints, leaving $6 - 3 = 3$ degrees of freedom. Since the anti-symmetric representation only admits 3 physical modes, it cannot accommodate the polarizations for a massive spin-2 particle, which has $2J + 1 = 5$ states by Wigner classification.

A momentum-dependent orthonormal basis tensors $\epsilon_A^{(i)\ \mu\nu}$ that satisfy the above constraints can be chosen as

$$\frac{1}{\sqrt{2}m}\begin{pmatrix} 0 & p_z & 0 & 0 \\ -p_z & 0 & 0 & -E \\ 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix}, \ \frac{1}{\sqrt{2}m}\begin{pmatrix} 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & 0 \\ -p_z & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, \ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8.48}$$

which satisfy the *anti-symmetric* and *transverse* conditions and are properly *normalized* via $\epsilon^{(i)\ \mu\nu}\epsilon^{*(j)}_{\mu\nu} = \delta^{ij}$, where $\epsilon_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}\epsilon^{\alpha\beta}$ happens to retain the same form as the original matrices under the Minkowski metric.

Similarly, the symmetric part, $\epsilon_S^{\mu\nu} \equiv M_S$, can be parametrized as:

$$\epsilon_S^{\mu\nu} \equiv M_S = \begin{pmatrix} s_{00} & s_{01} & s_{02} & s_{03} \\ s_{01} & s_{11} & s_{12} & s_{13} \\ s_{02} & s_{12} & s_{22} & s_{23} \\ s_{03} & s_{13} & s_{23} & s_{33} \end{pmatrix}. \tag{8.49}$$

Enforcing the transverse conditions $k_\mu \epsilon_S^{(i)\ \mu\nu} = 0$ leads to:

$$\begin{aligned} \nu = 0 :&\ \ Es_{00} - p_z s_{03} = 0, \\ \nu = 1 :&\ \ Es_{01} - p_z s_{13} = 0, \\ \nu = 2 :&\ \ Es_{02} - p_z s_{23} = 0, \\ \nu = 3 :&\ \ Es_{03} - p_z s_{33} = 0. \end{aligned} \tag{8.50}$$

A general symmetric matrix requires 10 parameters; the transversality conditions remove 4, leaving $10 - 4 = 6$ degrees of freedom. To isolate the $\mathbf{5}_S$, we seek a **Lorentz-invariant** condition to project out the singlet representation.

Naively, one might try to further decompose M_S into a traceless part $M_S - \mathrm{Tr}[M_S]\mathbb{I}$ and a trace-only part proportional to $\mathbb{I}$. However, the traceless condition $\sum_\mu \epsilon^{\mu\mu} = 0$ for a $(2,0)$ tensor is **not Lorentz-invariant**. **Hence, this condition can not remove a whole Lorentz representation.** The proper invariant constraint is to demand tracelessness of the associated $(1,1)$ tensor $\epsilon^\mu{}_\nu \equiv \epsilon^{\mu\alpha}g_{\alpha\nu}$. This implies that we actually need the condition

$$\epsilon_S^{\mu\alpha} g_{\alpha\mu} = s_{00} - s_{11} - s_{22} - s_{33} = 0. \tag{8.51}$$

This condition removes one additional dof, reducing the symmetric "traceless" space to $10 - 4 - 1 = 5$ degrees of freedom, while the "trace-only" part spans the singlet.

A basis for the symmetric traceless $\epsilon_S^{\mu\nu}$ space can be taken as:

$$\epsilon_1^{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_2^{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\epsilon_3^{\mu\nu} = \frac{1}{\sqrt{2}m} \begin{pmatrix} 0 & p_z & 0 & 0 \\ p_z & 0 & 0 & E \\ 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 \end{pmatrix}, \quad \epsilon_4^{\mu\nu} = \frac{1}{\sqrt{2}m} \begin{pmatrix} 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & 0 \\ p_z & 0 & 0 & E \\ 0 & 0 & E & 0 \end{pmatrix},$$
$$\epsilon_5^{\mu\nu} = \sqrt{\frac{2}{3}} \frac{1}{m^2} \begin{pmatrix} p_z^2 & 0 & 0 & p_z E \\ 0 & -\frac{1}{2}m^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}m^2 & 0 \\ p_z E & 0 & 0 & E^2 \end{pmatrix}, \tag{8.52}$$

each of which satisfies the *symmetric*, *"traceless"*, and *transverse* conditions, and is properly *normalized*.

The trivial singlet $\epsilon_{\mathbf{1}}^{\mu\nu}$ is proportional to the metric tensor $g^{\mu\nu}$ but cannot satisfy the transverse condition[3].

(c) Since the traceless symmetric, anti-symmetric, and "trace-only" components only transform among themselves under Lorentz transformations, each forms an irreducible representation of the Lorentz group. The previous analysis reveals that the "traceless" symmetric part has 5 degrees of freedom, corresponding to spin-2; the anti-symmetric part carries 3 degrees of freedom, corresponding to spin-1; and the trace-only part has a single degree of freedom, corresponding to spin-0.

Thus, to propagate the physical degrees of freedom of a spin-2 field, the polarization tensor must obey **symmetric**, **transverse**, and **"traceless"** conditions.

[3] More exactly, spin-0 does not have a well-defined notion of polarization.

Side Remark: What we have done are essentially the decomposition of the Lorentz group's irreducible representations into those of the rotation group, reflecting spin:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{5}_S \oplus \mathbf{3}_A \oplus \mathbf{1}.$$

In this sense, a spin-2 particle can be thought of as a "composite" of two spin-1 particles[a].
Indeed, one can construct the spin-2 polarization basis tensors directly by taking outer products of spin-1 polarization basis vectors. For instance, using the polarization basis from Eq. (8.30) and Eq. (8.31) of the textbook:

$$\epsilon_1^\mu = (0,1,0,0),\ \epsilon_2^\mu = (0,0,1,0),\ \epsilon_L^\mu = \frac{1}{m}(p_z,0,0,E). \tag{8.53}$$

The spin-2 polarization tensors in Eq. (8.52) can be written as:

$$\begin{aligned}
\epsilon_1^{\mu\nu} &= \frac{1}{\sqrt{2}}\left(\epsilon_1^\mu \bigotimes \epsilon_2^\nu + \epsilon_2^\mu \bigotimes \epsilon_1^\nu\right), \\
\epsilon_2^{\mu\nu} &= \frac{1}{\sqrt{2}}\left(\epsilon_1^\mu \bigotimes \epsilon_1^\nu - \epsilon_2^\mu \bigotimes \epsilon_2^\nu\right), \\
\epsilon_3^{\mu\nu} &= \frac{1}{\sqrt{2}}\left(\epsilon_1^\mu \bigotimes \epsilon_L^\nu + \epsilon_L^\mu \bigotimes \epsilon_1^\nu\right), \\
\epsilon_4^{\mu\nu} &= \frac{1}{\sqrt{2}}\left(\epsilon_2^\mu \bigotimes \epsilon_L^\nu + \epsilon_L^\mu \bigotimes \epsilon_2^\nu\right), \\
\epsilon_5^{\mu\nu} &= -\frac{1}{\sqrt{6}}\left(\epsilon_1^\mu \bigotimes \epsilon_1^\nu + \epsilon_2^\mu \bigotimes \epsilon_2^\nu - 2\epsilon_L^\mu \bigotimes \epsilon_L^\nu\right),
\end{aligned} \tag{8.54}$$

where the coefficients are exactly the Clebsch–Gordan coefficients. These are clearly symmetric under the exchange of the two sides of the outer products. The transverse conditions follow trivially because the spin-1 polarization basis vectors themselves obey transversality. Their tracelessness is evident, since the trace of an outer product of two vectors corresponds to their inner product (with respect to the metric tensor).

[a] Just suggesting its algebra, not its fundamentality.

(d) The basis for the physical polarization tensor corresponding to a massless spin-2 particle propagating along $k_\mu = (E,0,0,E)$ are the two tensors $\boxed{\epsilon_{1,2}^{\mu\nu}}$ appearing on the first line of Eq. (8.52). The remaining polarization tensors can not be properly normalized anymore, as their normalization products vanish.

Another way to identify which polarization tensors become ill-defined in the massless limit is by inspecting Eq. (8.54). Since, for a spin-1 particle, the longitudinal polarization $\epsilon_L^\mu \to p^\mu$ up to normalization becomes ill-defined in the massless limit and only $\epsilon_{1,2}^{\mu\nu}$ avoid involving ϵ_L^μ in their outer products, only $\epsilon_{1,2}^{\mu\nu}$ remain physical in the massless limit.

(e) For a spin-3 field, we embed it into a rank-3 tensor.

We shall impose the symmetric condition. In four-dimensional spacetime, this left with

$\binom{3+4-1}{3} = \frac{6!}{3!3!} = 20$ degrees of freedom.

Next, the transverse condition removes another 10 (since contraction with a four-momentum vector yields a symmetric rank-2 tensor, which has 10 degrees of freedom).

Finally, the "traceless" conditions (any contraction over two indices vanishes) remove 3 more degrees of freedom.

Altogether, we are left with $\boxed{20 - 10 - 3 = 7}$ physical degrees of freedom, precisely matching the expected count for a spin-3 field, which according to the Wigner classification has $2J + 1 = 2 \times 3 + 1 = 7$ degrees of freedom.

8.7

Suppose we now have a Lagrangian with a cubic interaction of h and undetermined coefficient a:

$$\mathcal{L}_4 = \left(1 + \frac{1}{2}h + \frac{1}{8}h^2 + ah^3\right)\phi. \tag{8.55}$$

Under the transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu + \pi^\alpha \partial_\alpha h_{\mu\nu} + (\partial_\mu \pi^\alpha) h_{\alpha\nu} + (\partial_\nu \pi^\alpha) h_{\mu\alpha}, \tag{8.56}$$

we have

$$h \to h + 2\partial_\nu \pi_\nu + \pi^\alpha \partial_\alpha h + 2(\partial_\mu \pi^\alpha) h_{\mu\alpha}, \tag{8.57}$$

and

$$\phi \to \phi + \pi^\alpha \partial_\alpha \phi. \tag{8.58}$$

The coefficient a is fixed by requiring cancellation of terms quadratic in h and linear in π. Thus, we only need to collect terms linear in π and at most $\mathcal{O}(h^2)$. The extra terms compared with Eq. (8.139) of the textbook are

$$\begin{aligned}\mathcal{L}_4 \to \mathcal{L}_4 &+ \frac{1}{4}h\pi_\alpha(\partial_\alpha h)\phi + \frac{1}{8}h^2\pi_\alpha(\partial_\alpha\phi) + (\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi + \frac{1}{2}h(\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi + 6ah^2(\partial_\alpha\pi_\alpha)\phi + \cdots \\ = \mathcal{L}_4 &- \frac{1}{8}h^2(\partial_\alpha\pi_\alpha)\phi + (\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi + \frac{1}{2}h(\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi + 6ah^2(\partial_\alpha\pi_\alpha)\phi + \cdots ,\end{aligned} \tag{8.59}$$

where $\cdots$ contain terms that are $\mathcal{O}(\pi^2)$ or $\mathcal{O}(h^3)$. Requiring cancellation of the h^2 terms gives

$$-\frac{1}{8}h^2(\partial_\alpha\pi_\alpha)\phi + 6ah^2(\partial_\alpha\pi_\alpha)\phi = 0 \tag{8.60}$$

$$\boxed{a = \frac{1}{48}}. \tag{8.61}$$

The terms $(\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi$ and $\frac{1}{2}h(\partial_\mu\pi^\alpha)h_{\mu\alpha}\phi$ remain because Eq. (8.138) of the textbook does not include all $\mathcal{O}(h_{\mu\nu}^2)$ interactions. To fix this, one must also add $-\frac{1}{4}(h_{\mu\nu})^2\phi$, and to obtain the full set of $\mathcal{O}(h_{\mu\nu}^3)$ interactions, include additional Lorentz-invariant cubic terms such as

$$bh(h_{\mu\nu})^2\phi + ch_{\mu\nu}h_{\nu\alpha}h_{\alpha\mu}\phi,$$

To see that explicitly, redo

$$\mathcal{L}_3' = \left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}^2\right)\phi. \tag{8.62}$$

The transformation Eq. (8.135) and Eq. (8.137) of the textbook are

$$\phi \to \phi + \pi_\nu \partial_\nu \phi, \tag{8.63}$$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu + \pi_\alpha \partial_\alpha h_{\mu\nu}. \tag{8.64}$$

We only write out the additional terms compared to the Eq. (8.139) of the textbook, and are at most $\mathcal{O}(\pi^1)$:

$$\begin{aligned}
\mathcal{L}_3' \to \mathcal{L}_3' &- \frac{1}{4}h_{\mu\nu}^2 \pi_\alpha \partial_\alpha \phi - \frac{1}{2}\phi h_{\mu\nu}(\partial_\mu \pi_\nu + \partial_\nu \pi_\mu + \pi_\alpha \partial_\alpha h_{\mu\nu}) + \cdots \\
&= \mathcal{L}_3' + \frac{1}{2}h_{\mu\nu}(\partial_\alpha h_{\mu\nu})\pi_\alpha \phi + \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi - \frac{1}{2}\phi h_{\mu\nu}(\partial_\mu \pi_\nu + \partial_\nu \pi_\mu + \pi_\alpha \partial_\alpha h_{\mu\nu}) + \cdots \\
&= \mathcal{L}_3' + \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi - \phi h_{\mu\nu}(\partial_\mu \pi_\nu) + \cdots,
\end{aligned} \tag{8.65}$$

where the last step follows from the fact that $-\frac{1}{2}\phi h_{\mu\nu}(\partial_\mu \pi_\nu + \partial_\nu \pi_\mu)$ is symmetric with respect to $\mu \leftrightarrow \nu$. Note that the last term cancels exactly one of the extra term in Eq. (8.59).

To determine b and c, we include higher-order transformations of $h_{\mu\nu}$ (cf. Eq. (8.140) of the textbook). The relevant terms are

$$\begin{aligned}
\mathcal{L}_4' \to \mathcal{L}_4' &+ \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi + \frac{1}{2}h(\partial_\mu \pi_\alpha)h_{\mu\alpha}\phi - \phi h_{\mu\nu}(\partial_\mu \pi_\alpha)h_{\alpha\nu} \\
&+ 2b\phi[h h_{\mu\nu}(\partial_\mu \pi_\nu + \partial_\nu \pi_\mu) + (\partial_\nu \pi_\nu)h_{\mu\nu}^2] \\
&\quad + c h_{\mu\nu}\phi[h_{\nu\alpha}(\partial_\alpha \pi_\mu + \partial_\mu \pi_\alpha) + (\partial_\nu \pi_\alpha + \partial_\alpha \pi_\nu)h_{\alpha\mu}] + c\phi(\partial_\mu \pi_\nu + \partial_\nu \pi_\mu)h_{\nu\alpha}h_{\alpha\mu} + \cdots \\
= \mathcal{L}_4' &+ \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi + \frac{1}{2}h(\partial_\mu \pi_\alpha)h_{\mu\alpha}\phi - \phi h_{\mu\nu}(\partial_\mu \pi_\alpha)h_{\alpha\nu} \\
&+ 2b\phi[2h h_{\mu\nu}(\partial_\mu \pi_\nu) + (\partial_\nu \pi_\nu)h_{\mu\nu}^2] + 2c h_{\mu\nu}\phi[h_{\nu\alpha}(\partial_\alpha \pi_\mu + \partial_\mu \pi_\alpha)] + 2c\phi(\partial_\mu \pi_\nu)h_{\nu\alpha}h_{\alpha\mu} + \cdots \\
= \mathcal{L}_4' &+ \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi + \frac{1}{2}h(\partial_\mu \pi_\alpha)h_{\mu\alpha}\phi - \phi h_{\mu\nu}(\partial_\mu \pi_\alpha)h_{\alpha\nu} \\
&+ 2b\phi[2h h_{\mu\nu}(\partial_\mu \pi_\nu) + (\partial_\nu \pi_\nu)h_{\mu\nu}^2] + 4c h_{\mu\nu}\phi h_{\nu\alpha}(\partial_\alpha \pi_\mu) + 2c\phi(\partial_\mu \pi_\nu)h_{\nu\alpha}h_{\alpha\mu} + \cdots \\
= \mathcal{L}_4' &+ \frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi + \frac{1}{2}h(\partial_\mu \pi_\alpha)h_{\mu\alpha}\phi - \phi h_{\mu\nu}(\partial_\mu \pi_\alpha)h_{\alpha\nu} \\
&+ 2b\phi[2h h_{\mu\nu}(\partial_\mu \pi_\nu) + (\partial_\nu \pi_\nu)h_{\mu\nu}^2] + 6c h_{\mu\nu}\phi h_{\nu\alpha}(\partial_\alpha \pi_\mu) + \cdots,
\end{aligned} \tag{8.66}$$

where I have repeatedly using the symmetric property under indices permutation. Now, we can observe that choosing

$$\boxed{b = -\frac{1}{8}} \tag{8.67}$$

ensures the cancellation of the terms

$$\frac{1}{4}h_{\mu\nu}^2(\partial_\alpha \pi_\alpha)\phi + \frac{1}{2}h(\partial_\mu \pi_\alpha)h_{\mu\alpha}\phi,$$

and choosing

$$\boxed{c = \frac{1}{6}} \tag{8.68}$$

ensures the cancellation of the term

$$-\phi h_{\mu\nu}(\partial_\mu \pi_\alpha) h_{\alpha\nu}.$$

Thus the interaction Lagrangian, up to cubic order in $h_{\mu\nu}$, is

$$\boxed{\mathcal{L} = \left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h^3 - \frac{1}{8}hh_{\mu\nu}h_{\mu\nu} + \frac{1}{6}h_{\mu\nu}h_{\nu\alpha}h_{\alpha\mu} + \mathcal{O}\left(h_{\mu\nu}^4\right)\right)\phi}\,. \tag{8.69}$$

Now, note that

$$h \equiv h_{\nu\nu} \equiv \mathrm{Tr}(h), \tag{8.70}$$

$$h_{\mu\nu}h_{\mu\nu} \equiv \mathrm{Tr}\left(h^2\right), \tag{8.71}$$

$$h_{\mu\nu}h_{\nu\alpha}h_{\alpha\mu} \equiv \mathrm{Tr}\left(h^3\right), \tag{8.72}$$

where the h in the trace is a matrix, it should not be confused with $h_{\nu\nu}$[4].

Using the identity

$$-\det\left(g_{\mu\nu}\right) = -\det\left(\eta_{\mu\nu} + h_{\mu\nu}\right) = e^{-\mathrm{Tr}\log(\eta+h)}. \tag{8.73}$$

and expanding for $h_{\mu\nu} \ll \eta_{\mu\nu}$ (weak-field approximation),

$$\log(\eta + h) = h - \frac{h^2}{2} + \frac{h^3}{3} + \mathcal{O}\left(h^4\right). \tag{8.74}$$

Eq. (8.145) of the textbook can be expanded as

$$\begin{aligned}
\mathcal{L} &= \sqrt{-\det\left(\eta_{\mu\nu} + h_{\mu\nu}\right)}\phi \\
&= \exp\left[-\frac{1}{2}\left(\mathrm{Tr}(h) - \frac{1}{2}\mathrm{Tr}\left(h^2\right) + \frac{1}{3}\mathrm{Tr}\left(h^3\right) + \mathrm{Tr}\left(\mathcal{O}\left(h^4\right)\right)\right)\right]\phi \\
&= \sqrt{-\det(\eta)}\left\{1 + \frac{1}{2}\mathrm{Tr}(h) - \frac{1}{4}\mathrm{Tr}\left(h^2\right) + \frac{1}{6}\mathrm{Tr}\left(h^3\right)\right. \\
&\quad \left. + \frac{1}{2!}\left[\left(\frac{1}{2}\mathrm{Tr}(h)\right)^2 - \frac{2}{8}\mathrm{Tr}(h)\,\mathrm{Tr}\left(h^2\right)\right] + \frac{1}{3!}\left(\frac{1}{2}\mathrm{Tr}(h)\right)^3 + \mathcal{O}\left(h^4\right)\right\}\phi \\
&= \left[1 + \frac{1}{2}\mathrm{Tr}(h) + \frac{1}{8}(\mathrm{Tr}(h))^2 - \frac{1}{4}\mathrm{Tr}\left(h^2\right)\right. \\
&\quad \left. + \frac{1}{48}(\mathrm{Tr}(h))^3 - \frac{1}{8}\mathrm{Tr}(h)\,\mathrm{Tr}\left(h^2\right) + \frac{1}{6}\mathrm{Tr}\left(h^3\right) + \mathcal{O}\left(h^4\right)\right]\phi \\
&= \boxed{\left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h^3 - \frac{1}{8}hh_{\mu\nu}h_{\mu\nu} + \frac{1}{6}h_{\mu\nu}h_{\nu\alpha}h_{\alpha\mu} + \mathcal{O}\left(h_{\mu\nu}^4\right)\right)\phi}\,,
\end{aligned} \tag{8.75}$$

where in the last line I wrote out the traces explicitly using Einstein summation (cf. Eq. (8.70)–(8.72)). This matches Eq. (8.69).

[4]Though unfortunately quite often share the same notation. I believe it should be clear that whenever the h appears in a Trace or log, I mean it as a matrix.

8.8

Starting from a totally symmetric rank-3 tensor $Z_{\mu\nu\alpha}$, decompose it as

$$Z_{\mu\nu\alpha} = Z^T_{\mu\nu\alpha} + \partial_\mu \pi_{\nu\alpha} + \partial_\nu \pi_{\alpha\mu} + \partial_\alpha \pi_{\mu\nu}, \tag{8.76}$$

with $\partial_\mu Z^T_{\mu\nu\alpha} = 0$. A massive spin-3 field has 7 polarizations: 2 in the transverse components and 5 in the longitudinal components $\pi_{\mu\nu}$. Since $Z_{\mu\nu\alpha}$ is totally symmetric, $\pi_{\mu\nu}$ is symmetric as well. We further decompose $\pi_{\mu\nu}$ as

$$\pi_{\mu\nu} = \pi^T_{\mu\nu} + \partial_\mu \sigma_\nu + \partial_\nu \sigma_\mu \tag{8.77}$$

with $\partial_\mu \pi^T_{\mu\nu} = 0$. Then decompose σ_μ as

$$\sigma_\mu = \sigma^T_\mu + \partial_\mu \sigma^L, \tag{8.78}$$

where $\partial_\mu \sigma^T_\mu = 0$.

The most general dimension-4 kinetic terms one can write for a rank-3 tensor $Z_{\mu\nu\alpha}$ are

$$\mathcal{L} = aZ_{\mu\nu\alpha}\Box Z_{\mu\nu\alpha} + bZ_{\mu\nu\alpha}\partial_\mu\partial_\beta Z_{\nu\beta\alpha} + cZ_\alpha \Box Z_\alpha + dZ_\alpha \partial_\mu\partial_\nu Z_{\mu\nu\alpha} + eZ_\mu \partial_\mu\partial_\nu Z_\nu + m^2(xZ^2_{\mu\nu\alpha} + yZ^2_\alpha), \tag{8.79}$$

where $Z_\alpha \equiv Z_{\mu\mu\alpha} = Z_{\mu\alpha\mu} = Z_{\alpha\mu\mu}$ is the partial trace of the tensor. Also,

$$\pi = \pi^T + 2\partial_\mu \sigma_\mu = \pi^T + 2\Box\sigma^L, \tag{8.80}$$

$$Z_\alpha \equiv Z_{\mu\mu\alpha} = Z^T_\alpha + 2\partial_\mu \pi_{\mu\alpha} + \partial_\alpha \pi = Z^T_\alpha + 2\Box\sigma_\alpha + 2\partial_\alpha\partial_\mu\sigma_\mu + \partial_\alpha \pi^T + 2\Box\partial_\alpha\sigma^L = Z^T_\alpha + \partial_\alpha\pi^T + 2(\Box\sigma^T_\alpha + 3\Box\partial_\alpha\sigma^L). \tag{8.81}$$

Consider the mass term first:

$$\begin{aligned} Z^2_{\mu\nu\alpha} &= (Z^T_{\mu\nu\alpha} + \partial_\mu\pi_{\nu\alpha} + \partial_\nu\pi_{\alpha\mu} + \partial_\alpha\pi_{\mu\nu})^2 \\ &= 4(\partial_\mu\partial_\nu\sigma_\alpha + \partial_\mu\partial_\alpha\sigma_\nu + \partial_\nu\partial_\alpha\sigma_\mu)^2 + \cdots \\ &= 4(\partial_\mu\partial_\nu\sigma^T_\alpha + \partial_\mu\partial_\alpha\sigma^T_\nu + \partial_\nu\partial_\alpha\sigma^T_\mu + 3\partial_\mu\partial_\nu\partial_\alpha\sigma^L)^2 + \cdots \\ &= 4(\partial_\mu\partial_\nu\sigma^T_\alpha)^2 + 4(\partial_\mu\partial_\alpha\sigma^T_\nu)^2 + 4(\partial_\nu\partial_\alpha\sigma^T_\mu)^2 + 36(\partial_\mu\partial_\nu\partial_\alpha\sigma^L)^2 + \cdots \\ &= 12\sigma^T_\alpha\Box^2\sigma^T_\alpha - 108\sigma^L\Box^3\sigma^L + \cdots, \end{aligned} \tag{8.82}$$

where $\cdots$ contain terms with no more than two derivatives. I repeatedly used $\partial_\mu\pi^T_{\mu\nu} = 0$ and $\partial_\mu\sigma^T_\mu = 0$, and integrated by parts in the last line. Note the extra factor $\frac{3!}{2!1!} = 3$ when contracting $(\partial_\mu\partial_\nu\partial_\alpha\sigma^L)^2$ arises because the tensor is totally symmetric.

Next,

$$\begin{aligned} Z^2_\alpha &= (Z^T_\alpha)^2 - \pi^T\Box\pi^T + 4Z^T_\alpha\Box\sigma^T_\alpha - 12\pi^T\Box^2\sigma^L + 4(\Box\sigma^T_\alpha + 3\Box\partial_\alpha\sigma^L)^2 \\ &= -12\pi^T\Box^2\sigma^L + 4(\Box\sigma^T_\alpha)^2 + 36(\Box\partial_\alpha\sigma^L)^2 + \cdots \\ &= -12\pi^T\Box^2\sigma^L + 4\sigma^T_\alpha\Box^2\sigma^T_\alpha - 36\sigma^L\Box^3\sigma^L + \cdots. \end{aligned} \tag{8.83}$$

Comparing the two mass terms and requiring cancellation of the dangerous four-derivative (or higher) pieces,

$$\begin{cases} \sigma^T_\alpha\Box^2\sigma^T_\alpha : & m^2(12x + 4y) = 0 \\ \sigma^L\Box^3\sigma^L : & m^2(-108x - 36y) = 0 \end{cases} \implies \boxed{y = -3x}, \tag{8.84}$$

and

$$-12m^2 y\pi^T\Box^2\sigma^L = 0 \implies \boxed{\pi^T = 0}\,, \tag{8.85}$$

for a nontrivial mass term, indicating that $\pi^T_{\mu\nu}$ must be traceless, as expected (confirming Problem 8.6).

Now turn to the other terms. For later use, first compute

$$\begin{aligned}\partial_\mu Z_{\mu\nu\alpha} &= \Box\pi_{\nu\alpha} + \partial_\mu\partial_\nu\pi_{\alpha\mu} + \partial_\mu\partial_\alpha\pi_{\mu\nu}\\ &= \Box\pi^T_{\nu\alpha} + 2(\Box\partial_\nu\sigma_\alpha + \Box\partial_\alpha\sigma_\nu + \partial_\mu\partial_\nu\partial_\alpha\sigma_\mu)\\ &= \Box\pi^T_{\nu\alpha} + 2(\Box\partial_\nu\sigma^T_\alpha + \Box\partial_\alpha\sigma^T_\nu + 3\Box\partial_\nu\partial_\alpha\sigma^L).\end{aligned} \tag{8.86}$$

Also,

$$\partial_\mu\partial_\nu Z_{\mu\nu\alpha} = 2(\Box^2\sigma^T_\alpha + 3\Box^2\partial_\alpha\sigma^L), \tag{8.87}$$

and

$$\partial_\mu Z_\mu = 6\Box^2\sigma^L \tag{8.88}$$

- $\boldsymbol{Z_{\mu\nu\alpha}\Box Z_{\mu\nu\alpha}}$**:**

$$\begin{aligned}Z_{\mu\nu\alpha}\Box Z_{\mu\nu\alpha} &= (Z^T_{\mu\nu\alpha} + \partial_\mu\pi_{\nu\alpha} + \partial_\nu\pi_{\alpha\mu} + \partial_\alpha\pi_{\mu\nu})\Box(Z^T_{\mu\nu\alpha} + \partial_\mu\pi_{\nu\alpha} + \partial_\nu\pi_{\alpha\mu} + \partial_\alpha\pi_{\mu\nu})\\ &= Z^T_{\mu\nu\alpha}\Box Z^T_{\mu\nu\alpha} + (\partial_\mu\pi_{\nu\alpha} + \partial_\nu\pi_{\alpha\mu} + \partial_\alpha\pi_{\mu\nu})\Box(\partial_\mu\pi_{\nu\alpha} + \partial_\nu\pi_{\alpha\mu} + \partial_\alpha\pi_{\mu\nu})\\ &= Z^T_{\mu\nu\alpha}\Box Z^T_{\mu\nu\alpha} + [\partial_\mu\pi^T_{\nu\alpha} + \partial_\nu\pi^T_{\alpha\mu} + \partial_\alpha\pi^T_{\mu\nu} + 2(\partial_\mu\partial_\nu\sigma_\alpha + \partial_\mu\partial_\alpha\sigma_\nu + \partial_\nu\partial_\alpha\sigma_\mu)]\Box[\cdots]\\ &= Z^T_{\mu\nu\alpha}\Box Z^T_{\mu\nu\alpha} - 3\pi^T_{\nu\alpha}\Box^2\pi^T_{\nu\alpha} + [2(\partial_\mu\partial_\nu\sigma^T_\alpha + \partial_\mu\partial_\alpha\sigma^T_\nu + \partial_\nu\partial_\alpha\sigma^T_\mu + 3\partial_\mu\partial_\nu\partial_\alpha\sigma^L)]\Box[\cdots]\\ &= Z^T_{\mu\nu\alpha}\Box Z^T_{\mu\nu\alpha} - 3\pi^T_{\nu\alpha}\Box^2\pi^T_{\nu\alpha} + 12\sigma^T_\alpha\Box^3\sigma^T_\alpha - 108\sigma^L\Box^4\sigma^L.\end{aligned} \tag{8.89}$$

- $\boldsymbol{Z_{\mu\nu\alpha}\partial_\mu\partial_\beta Z_{\nu\beta\alpha}}$**:**

$$\begin{aligned}Z_{\mu\nu\alpha}\partial_\mu\partial_\beta Z_{\nu\beta\alpha} &= -(\partial_\mu Z_{\mu\nu\alpha})^2\\ &= -[\Box\pi^T_{\nu\alpha} + 2(\Box\partial_\nu\sigma^T_\alpha + \Box\partial_\alpha\sigma^T_\nu + 3\Box\partial_\nu\partial_\alpha\sigma^L)]^2\\ &= -\pi^T_{\nu\alpha}\Box^2\pi^T_{\nu\alpha} + 8\sigma^T_\alpha\Box^3\sigma^T_\alpha - 36\sigma^L\Box^4\sigma^L.\end{aligned} \tag{8.90}$$

- $\boldsymbol{Z_\alpha\Box Z_\alpha}$**:**

$$\begin{aligned}Z_\alpha\Box Z_\alpha &= [Z^T_\alpha + 2(\Box\sigma^T_\alpha + 3\Box\partial_\alpha\sigma^L)]\Box[Z^T_\alpha + 2(\Box\sigma^T_\alpha + 3\Box\partial_\alpha\sigma^L)]\\ &= Z^T_\alpha\Box Z^T_\alpha + 4Z^T_\alpha\Box^2\sigma^T_\alpha + 4\sigma^T_\alpha\Box^3\sigma^T_\alpha - 36\sigma^L\Box^4\sigma^L.\end{aligned} \tag{8.91}$$

- $\boldsymbol{Z_\alpha\partial_\mu\partial_\nu Z_{\mu\nu\alpha}}$**:**

$$\begin{aligned}Z_\alpha\partial_\mu\partial_\nu Z_{\mu\nu\alpha} &= 2[Z^T_\alpha + 2(\Box\sigma^T_\alpha + 3\Box\partial_\alpha\sigma^L)](\Box^2\sigma^T_\alpha + 3\Box^2\partial_\alpha\sigma^L)\\ &= 2Z^T_\alpha\Box^2\sigma^T_\alpha + 4\sigma^T_\alpha\Box^3\sigma^T_\alpha - 36\sigma^L\Box^4\sigma^L.\end{aligned} \tag{8.92}$$

- $\boldsymbol{Z_\mu\partial_\mu\partial_\nu Z_\nu}$ **:**

$$\begin{aligned}Z_\mu\partial_\mu\partial_\nu Z_\nu &= -2(\partial_\mu Z_\mu)^2\\ &= -72\sigma^L\Box^4\sigma^L,\end{aligned} \tag{8.93}$$

where the extra factor of 2 arises because there are two inequivalent ways to take the partial traces on the two sides.

Requiring cancellation of the dangerous four-derivative (or higher) terms,

$$\boxed{\begin{cases} \pi^T_{\nu\alpha}\Box^2\pi^T_{\nu\alpha}: & -3a-b=0 \\ \sigma^T_\alpha\Box^3\sigma^T_\alpha: & 12a+8b+4c+4d=0 \\ \sigma^L\Box^4\sigma^L: & -108a-36b-36c-36d-72e=0 \\ Z^T_\alpha\Box^2\sigma^T_\alpha: & 4c+2d=0 \end{cases} \implies \begin{cases} b=-3a \\ c=-3a \\ d=-2c \\ c=2e \end{cases}}. \tag{8.94}$$

Fix the overall normalization by imposing the equation of motion for $Z^T_{\mu\nu\alpha}$,

$$(\Box+m^2)Z^T_{\mu\nu\alpha}=0,$$

and require positive energy. These fix

$$\boxed{a=\frac{1}{2}, b=-\frac{3}{2}, c=-\frac{3}{2}, d=3, e=-\frac{3}{4}, x=\frac{1}{2}, y=-\frac{3}{2}}. \tag{8.95}$$

Hence,

$$\boxed{\mathcal{L}=\frac{1}{2}Z_{\mu\nu\alpha}\Box Z_{\mu\nu\alpha}-\frac{3}{2}Z_{\mu\nu\alpha}\partial_\mu\partial_\beta Z_{\nu\beta\alpha}-\frac{3}{2}Z_\alpha\Box Z_\alpha+3Z_\alpha\partial_\mu\partial_\nu Z_{\mu\nu\alpha}-\frac{3}{4}Z_\mu\partial_\mu\partial_\nu Z_\nu+\frac{1}{2}m^2(Z^2_{\mu\nu\alpha}-3Z^2_\alpha)}. \tag{8.96}$$

8.9

As derived in Problem 3.1, Eq. (3.1), the generalized equation of motion for a Lagrangian with up to two derivatives is

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)+\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\phi)}\right)=0. \tag{8.97}$$

Starting from the Lagrangian

$$\mathcal{L}=-a\phi\Box\phi-b\phi\Box^2\phi=a(\partial_\mu\phi)^2-b(\Box\phi)^2, \tag{8.98}$$

for arbitrary constants a and b. We ignore the mass term, which is a trivial extension. The ill behavior of such a Lagrangian is already evident from its equation of motion:

$$\Box(a+b\Box)\phi=0 \implies a\Box\phi=-b\Box^2\phi. \tag{8.99}$$

One can repeatedly apply the equation of motion, relating the two-derivative piece of the field to arbitrarily high derivatives.

We now generalize Noether's theorem to higher derivatives. Much of the setup follows Problem 3.1. Under global spacetime translations $\phi(x)\to\phi(x+\xi)$ with infinitesimal ξ^ν, and using Eq. (3.1), the on-shell variation of the Lagrangian (for terms up to second derivatives of ϕ) isfield ϕ is

$$\frac{\delta\mathcal{L}}{\delta\xi^\nu}=\partial_\mu\left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\xi^\nu}\right)+2\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\alpha\phi)}\frac{\delta(\partial_\alpha\phi)}{\delta\xi^\nu}\right)-\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\alpha\phi)}\frac{\delta\phi}{\delta\xi^\nu}\right)\right]. \tag{8.100}$$

Here $\frac{\delta\phi}{\delta\xi^\nu} = \partial_\nu\phi$ and $\frac{\delta(\partial_\alpha\phi)}{\delta\xi^\nu} = \partial_\nu\partial_\alpha\phi$.

Since $\mathcal{L}$ is itself a scalar under translations, $\mathcal{L}(x) \to \mathcal{L}(x+\xi)$, we also have

$$\frac{\delta\mathcal{L}}{\delta\xi^\nu} = \partial_\nu\mathcal{L}. \tag{8.101}$$

Equating the two expressions yields the energy–momentum tensor

$$\mathcal{T}_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + 2\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\alpha\phi)}\partial_\nu\partial_\alpha\phi - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\alpha\phi)}\partial_\nu\phi\right) - g_{\mu\nu}\mathcal{L} \tag{8.102}$$

Plugging Eq. (8.98) into this,

$$\begin{aligned}
\mathcal{T}_{\mu\nu} &= 2a(\partial_\mu\phi)(\partial_\nu\phi) - 4b(\Box\phi)\frac{\partial(\partial_\beta\partial_\beta\phi)}{\partial(\partial_\mu\partial_\alpha\phi)}\partial_\nu\partial_\alpha\phi + 2b\partial_\alpha\left((\Box\phi)\frac{\partial(\partial_\beta\partial_\beta\phi)}{\partial(\partial_\mu\partial_\alpha\phi)}\partial_\nu\phi\right) + g_{\mu\nu}[a(\partial_\beta\phi)^2 - b(\Box\phi)^2] \\
&= 2a(\partial_\mu\phi)(\partial_\nu\phi) - 4b(\Box\phi)g_{\beta\mu}g_{\beta\alpha}\partial_\nu\partial_\alpha\phi + 2b\partial_\alpha\left[(\Box\phi)g_{\beta\mu}g_{\beta\alpha}\partial_\nu\phi\right] - g_{\mu\nu}[a(\partial_\beta\phi)^2 - b(\Box\phi)^2] \\
&= 2a(\partial_\mu\phi)(\partial_\nu\phi) - 4b(\Box\phi)\partial_\nu\partial_\mu\phi + 2b\partial_\mu\left[(\Box\phi)\partial_\nu\phi\right] - g_{\mu\nu}[a(\partial_\beta\phi)^2 - b(\Box\phi)^2] \\
&= 2a(\partial_\mu\phi)(\partial_\nu\phi) - 2b(\Box\phi)\partial_\nu\partial_\mu\phi + 2b(\partial_\mu\Box\phi)(\partial_\nu\phi) - g_{\mu\nu}[a(\partial_\beta\phi)^2 - b(\Box\phi)^2].
\end{aligned} \tag{8.103}$$

The energy density is

$$\begin{aligned}
\mathcal{E} &= \mathcal{T}_{00} \\
&= a\left[(\partial_t\phi)^2 + (\nabla\phi)^2\right] + b\left[-2(\Box\phi)(\partial_t^2\phi) + 2(\partial_t\Box\phi)(\partial_t\phi) + (\Box\phi)^2\right] \\
&= a\left[(\partial_t\phi)^2 + (\nabla\phi)^2\right] \\
&\quad + b\left[-2(\partial_t^2\phi)^2 + 2(\nabla^2\phi)(\partial_t^2\phi) + 2(\partial_t^3\phi)(\partial_t\phi) - 2(\partial_t\nabla^2\phi)(\partial_t\phi) + (\partial_t^2\phi)^2 + (\nabla^2\phi)^2 - 2(\partial_t^2\phi)(\nabla^2\phi)\right] \\
&= \boxed{a[(\partial_t\phi)^2 + (\nabla\phi)^2] + b[-(\partial_t^2\phi)^2 + 2(\partial_t^3\phi)(\partial_t\phi) - 2(\partial_t\nabla^2\phi)(\partial_t\phi) + (\nabla^2\phi)^2]}\,,
\end{aligned} \tag{8.104}$$

which contains unavoidable negative contributions. Unlike the case of a spin-1 vector field, a scalar has no gauge redundancy to impose additional constraints, so the ghost term cannot be removed.

Chapter 9

Scalar quantum electrodynamics

9.1

We use p_1 to denote the momentum of the incoming photon, p_2 that of the incoming scalar, p_3 that of the outgoing scalar, and p_4 that of the outgoing photon.

(a) The s-channel gives

$$i\mathcal{M}_s = -ie^2 \frac{(p_1 \cdot \epsilon_1 + 2p_2 \cdot \epsilon_1)(p_4 \cdot \epsilon_4^* + 2p_3 \cdot \epsilon_4^*)}{p_1^2 + 2p_1 \cdot p_2}, \tag{9.1}$$

where we use the fact that the electrons are on-shell. The t-channel gives

$$i\mathcal{M}_t = -ie^2 \frac{(p_1 \cdot \epsilon_1 - 2p_3 \cdot \epsilon_1)(p_4 \cdot \epsilon_4^* - 2p_2 \cdot \epsilon_4^*)}{p_1^2 - 2p_1 \cdot p_3} \tag{9.2}$$

Lastly, the seagull vertex gives

$$i\mathcal{M}_4 = 2ie^2 g_{\mu\nu} \epsilon_1^\mu \epsilon_4^{*\nu} \tag{9.3}$$

To check Ward identity, we replace ϵ_1^μ with p_1^μ and summing all the diagrams, we have

$$\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_4 = -e^2 \epsilon_4^{*\mu} (p_4 + 2p_3 + p_4 - 2p_2 - 2p_1)^\mu = 0, \tag{9.4}$$

and the Ward identity is satisfied.

Now using the fact that the physical polarizations of the photon must be orthogonal to the photon's momentum, we have $p_1 \cdot \epsilon_1 = p_4 \cdot \epsilon_4^* = 0$. Furthermore, in CM frame, we have $p_1 = -p_2$ and $p_3 = -p_4$ such that $p_2 \cdot \epsilon_1 = p_3 \cdot \epsilon_4^* = 0$. We also know the on-shell photon is massless such that $p_1^2 = p_4^2 = 0$. We can thus simplify the matrix element such that

$$\begin{aligned} i\mathcal{M}_{tot} &= i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_4 \\ &= 2ie^2 \left[\frac{(p_3 \cdot \epsilon_1)(p_2 \cdot \epsilon_4^*)}{p_1 \cdot p_3} + \epsilon_1 \cdot \epsilon_4^* \right] \end{aligned} \tag{9.5}$$

(b) In the CM frame, $p_1 = (E_1, 0, 0, E_1)$, $p_2 = (E_2, 0, 0, -E_1)$, $p_3 = (E_2, -E_1 \sin\theta, 0, -E_1 \cos\theta)$, and $p_4 = (E_1, E_1 \sin\theta, 0, E_1 \cos\theta)$, where $E_2 = \sqrt{E_1^2 + m_\phi^2}$ and θ is the angle between the incoming scalar and the outgoing photon. We also have $\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} |\mathcal{M}|^2$

(c) For ϵ_μ^{in} polarized in the plane of the scattering, we choose the basis $\epsilon_\mu^{in} = (0,1,0,0)$. For the outgoing polarization, we choose $\epsilon_\mu^{out1*} = (0, \cos\theta, 0, -\sin\theta)$, polarized in the plane of the scattering, and $\epsilon_\mu^{out2*} = (0,0,1,0)$, polarized transverse to the plane of the scattering.

$$\begin{aligned}
|\mathcal{M}_{in}|^2 &= 4e^4\left(\left|\frac{(p_3\cdot\epsilon^{in})(p_2\cdot\epsilon^{out1*})}{p_1\cdot p_3} + \epsilon^{in}\cdot\epsilon^{out1*}\right|^2 + \left|\frac{(p_3\cdot\epsilon^{in})(p_2\cdot\epsilon^{out2*})}{p_1\cdot p_3} + \epsilon^{in}\cdot\epsilon^{out2*}\right|^2\right)\\
&= 4e^4\left(\frac{(E_1\sin\theta)(-E_1\sin\theta)}{E_1E_2 + E_1^2\cos\theta} - \cos\theta\right)^2 + 0\\
&= 4e^4\left(\frac{E_1 + E_2\cos\theta}{E_2 + E_1\cos\theta}\right)^2
\end{aligned} \tag{9.6}$$

(d) For ϵ_μ^{in} polarized transverse to the plane of the scattering, we choose the basis $\epsilon_\mu^{in} = (0,0,1,0)$. Follow the same calculations as in last part, we have

$$|\mathcal{M}_{transverse}|^2 = 4e^4 \tag{9.7}$$

(e) Summing up the (c) and (d), we have

$$|\mathcal{M}|^2 = 4e^4\left(1 + (\frac{E_1 + E_2\cos\theta}{E_2 + E_1\cos\theta})^2\right) \tag{9.8}$$

Doing the replacement from part (a), we have

$$\begin{aligned}
|\mathcal{M}|^2 &= |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + |\mathcal{M}_4|^2 + \mathcal{M}_s\mathcal{M}_t^* + \mathcal{M}_t\mathcal{M}_s^* + \mathcal{M}_s\mathcal{M}_4^* + \mathcal{M}_4\mathcal{M}_s^* + \mathcal{M}_4\mathcal{M}_t^* + \mathcal{M}_t\mathcal{M}_4^*\\
&= e^4[\frac{(p_1+2p_2)^2(p_4+2p_3)^2}{(2p_1\cdot p_2)^2} + \frac{(p_1-2p_3)^2(p_4-2p_2)^2}{(2p_1\cdot p_3)^2} + 16\\
&+ 2\frac{(p_1+2p_2)\cdot(p_1-2p_3)(p_4+2p_3)\cdot(p_4-2p_2)}{(2p_1\cdot p_2)(-2p_1\cdot p_3)} + 8\frac{(p_1+2p_2)\cdot(p_4+2p_3)}{(2p_1\cdot p_2)}\\
&+ 8\frac{(p_1-2p_3)\cdot(p_4-2p_2)}{(-2p_1\cdot p_3)}]\\
&= e^4[\frac{4E_2^2}{E_1^2} + \frac{4(m^2 - E_1(E_2+E_1\cos\theta))^2}{E_1^2(E_2+E_1\cos\theta)^2} + 16 - \frac{2(E_1^2(1-\cos\theta) - 2(E_2^2 - E_1^2\cos\theta))^2}{E_1^2(E_1+E_2)(E_2+E_1\cos\theta)}]\\
&- \frac{2(4E_2^2 + E_1^2(1-5\cos\theta) - 4E_1(E_1+E_2\cos\theta))}{E_1(E_1+E_2)} + \frac{2(4E_2^2 + E_1^2(1-5\cos\theta) - 4E_1(E_1+E_2))}{E_1(E_2+E_1\cos\theta)}\\
&= \frac{e^4}{E_1^2(E_2+E_1)^2(E_2+E_1\cos\theta)^2}(4E_1^2(E_1+E_2)^2((E_1+E_2\cos\theta)^2 + (E_2+E_1\cos\theta)^2))\\
&= \frac{4e^4}{(E_2+E_1\cos\theta)^2}((E_1+E_2\cos\theta)^2 + (E_2+E_1\cos\theta)^2)
\end{aligned} \tag{9.9}$$

This is the same as equation 9.8. Thus, the replacement trick works.

(f) Such replacement trick only works for a Abelian massless spin-1 particle. Also, notice that to use such replacement, we must include the unphysical polarization as well. This is the reason why we must do the replacement from part (a).

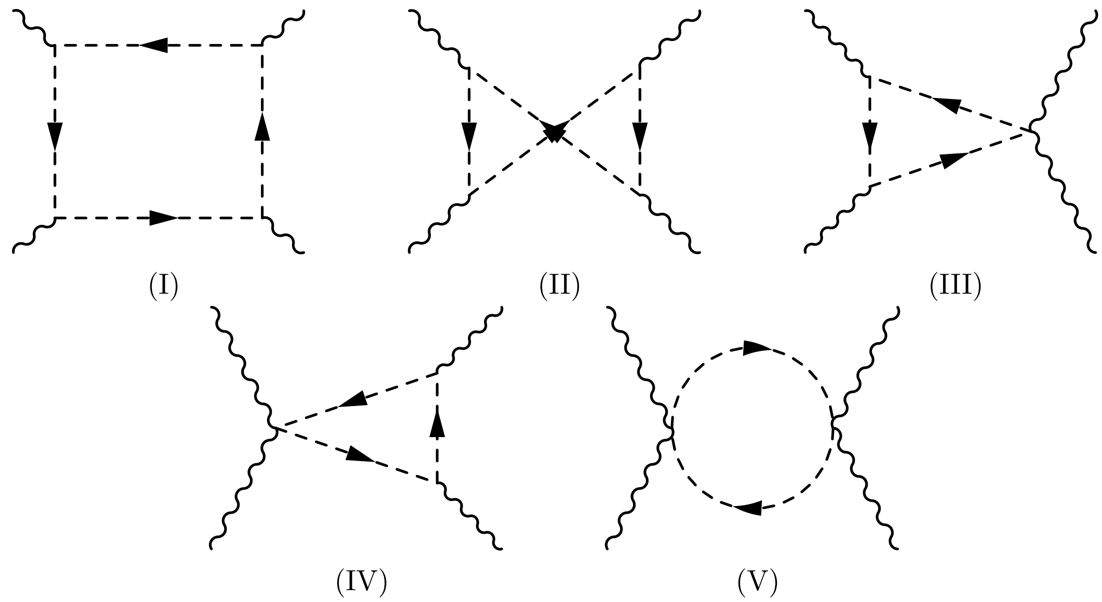


Fig. 9.1: $\mathcal{O}(e^4)$ light-by-light scattering diagrams

9.2

(a) This is a $\mathcal{O}(e^8)$ diagram. We shall start the counting from $\mathcal{O}(e^4)$ first since $\mathcal{O}(e^4)$ is the lowest order of light-by-light scattering happens. At $\mathcal{O}(e^4)$, there are the following 5 diagrams listed in Fig. 9.1:

Now let's consider the diagrams at $\mathcal{O}(e^6)$. First notice that for each of the photon line, it's possible to have the following vacuum polarization correction diagrams shown in Fig. 18.1 (i) and (ii) that each itself is at $\mathcal{O}(e^2)$. Similarly, for each of the scalar line, we can attach the correction as Fig. 18.1 (iii), (iv) and (v). For each of the 3-point vertex, we can attach with the tadpole in Fig. 18.1 (vi), where the shaded dot means a 3-point vertex. Then, there are also corrections of which a photon line either starts from a scalar line or a 3-point vertex and ends at either an another scalar line or an another 3-point vertex as shown in Fig. 18.1 (vii), (viii), and (xi), where the big shaded dot refers to the rest part of the diagram that we don't draw and the small shaded dot refers to a 3-point vertex. Notice for (viii) and (xi), it's not necessary that the scalar line must start from the same vertex as the photon line. (viii) should be interpreted as starting from a 3-point-vertex and ending at a scalar line while (xi) should be interpreted as starting from a 3-point vertex and ending at another 3-point vertex.

Now we can do the do the counting. Starting from the diagram I. To modify it to order $\mathcal{O}(e^6)$, we can do the following modifications: for each of the 4 photon lines, we can either modify it with diagram (i) or (ii); for each of the 4 scalar lines, we can either modify it with diagram (iii) or (iv) or (v); for each of the 4 3-point vertexes, we can attach them with the tadpole diagram (vi); for any two of the 4 scalar lines, we can connect them with a photon line ((vii)); we can connect anyone of the 4 3-point vertexes to anyone of the 4 scalar lines with a photon line ((viii)); we can connect any two of the 4 3-point vertexes with a photon

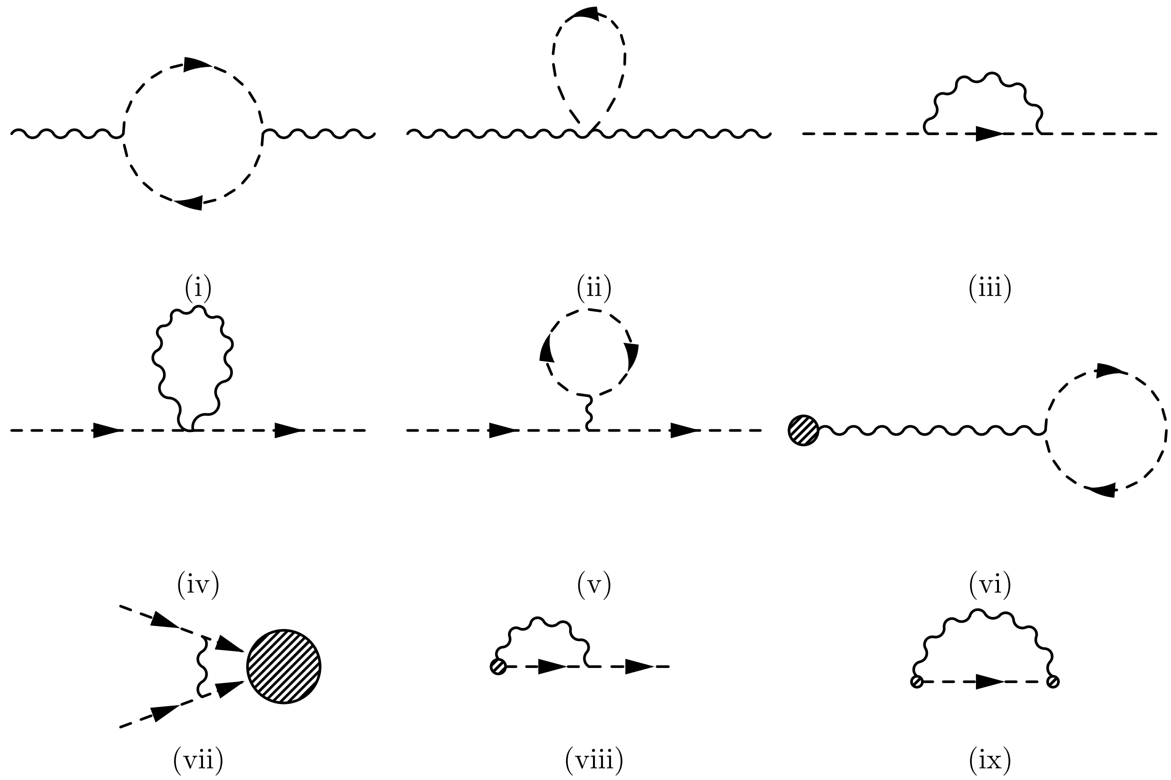


Fig. 9.2: $\mathcal{O}(e^2)$ correction

line ((xi)). We summarize these operations we can do with the diagram I in Eq. (9.10).

$$
\begin{aligned}
&4P - (i)/(ii):\ 4 \times 2 = 8,\\
&4S - (iii)(iv)(v):\ 4 \times 3 = 12,\\
&4V - (vi):\ 4,\\
&4S - (vii):\ C_2^4 = \binom{4}{2} = \frac{4!}{2!2!} = 6,\\
&4V - (viii) - 4S:\ 4 \times 4 = 16,\\
&4V - (xi):\ C_2^4 = \binom{4}{2} = \frac{4!}{2!2!} = 6,\\
&\text{total: } 8 + 12 + 4 + 6 + 16 + 6 = 52,
\end{aligned}
\tag{9.10}
$$

where P refers to a photon line, S refers to a scalar line, and V refers to a 3-point vertex. We thus conclude that for $\mathcal{O}(e^6)$ correction from diagram I, there are 52 diagrams at $\mathcal{O}(e^6)$. Clearly, the diagram II just flips the final states compared with the diagram I and should also have 52 diagrams that can have $\mathcal{O}(e^6)$ correction. For diagram III and IV, we again

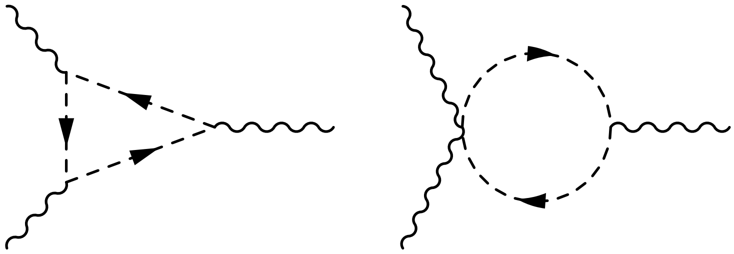


Fig. 9.3: $\mathcal{O}(e^3)$ diagrams with 3 external photon lines

can summarize the operations as in Eq. (9.11).

$$\begin{aligned}
&4P-(i)/(ii):\ 4\times 2=8,\\
&3S-(iii)(iv)(v):\ 3\times 3=9,\\
&2V-(vi):\ 2,\\
&3S-(vii):\ C_2^3=\binom{3}{2}=\frac{3!}{2!1!}=3,\\
&2V-(viii)-3S:\ 2\times 3=6,\\
&2V-(xi):\ C_2^2=\binom{2}{2}=1,\\
&\text{total: } 8+9+2+3+6+1=29,
\end{aligned} \tag{9.11}$$

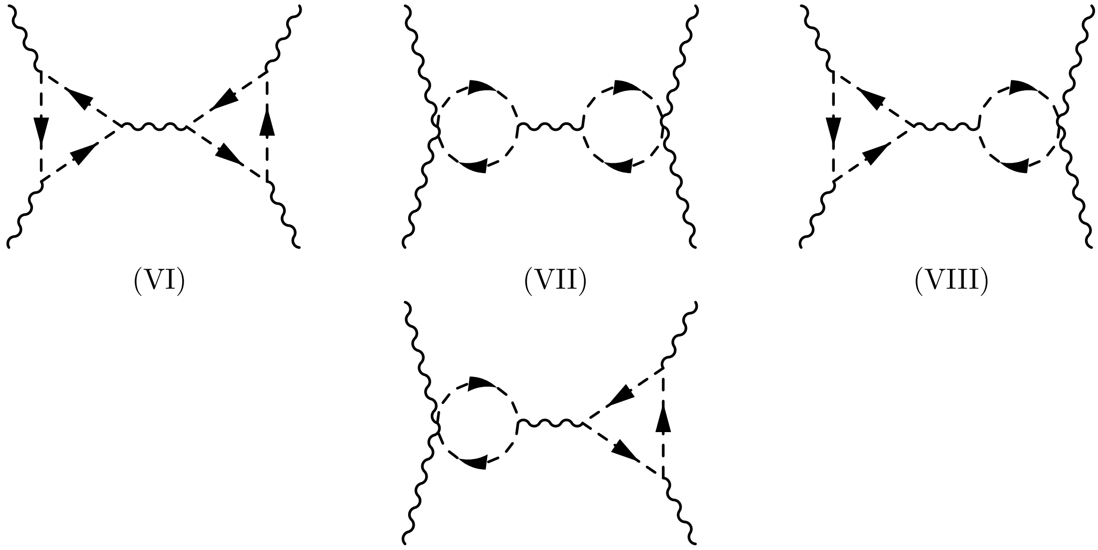
Thus, the diagram III (IV) shall have 29 diagrams with correction at $\mathcal{O}(e^6)$. For the diagram V, we summarize the operations in Eq. (9.12).

$$\begin{aligned}
&4P-(i)/(ii):\ 4\times 2=8,\\
&2S-(iii)(iv)(v):\ 2\times 3=6,\\
&0V-(vi):\ 0,\\
&2S-(vii):\ C_2^2=\binom{2}{2}=1,\\
&0V-(viii)-2S:\ 0,\\
&0V-(xi):\ 0.\\
&\text{total: } 8+6+1=15,
\end{aligned} \tag{9.12}$$

Thus, the diagram V have 15 diagrams with correction at $\mathcal{O}(e^6)$. Lastly, there are also extra diagrams, which can not be gotten from combining an $\mathcal{O}(e^4)$ diagram with an $\mathcal{O}(e^2)$ diagram but from combining two $\mathcal{O}(e^3)$ diagrams like the ones in Fig. 9.3 and got 4 extra diagrams at $\mathcal{O}(e^6)$ as shown in Fig. 9.4.

In total, at $\mathcal{O}(e^6)$, we have $52\times 2+29\times 2+15+4=181$ diagrams.

We can continue this way of counting to $\mathcal{O}(e^8)$. First, we shall notice the number of lines



(VI) (VII) (VIII)

(IX)

Fig. 9.4: $\mathcal{O}(e^6)$ light-by-light scattering diagrams by combining any two $\mathcal{O}(e^3)$ diagrams in Fig. 9.3.

or 3-point vertexes added by above operations are summarized in Eq. (9.13).

$$\begin{aligned}
&(i) : +2S + 1P + 2V + 0V',\\
&(ii) : +1S + 1P + 0V + 1V',\\
&(iii) : +2S + 1P + 2V + 0V',\\
&(iv) : +1S + 1P + 0V + 1V',\\
&(v) : +2S + 1P + 2V + 0V',\\
&(vi) : +1S + 1P - 1V + 1V',\\
&(vii) : +2S + 1P + 2V + 0V',\\
&(viii) : +1S + 1P + 0V + 1V',\\
&(xi) : +0S + 1P - 2V + 2V'.
\end{aligned} \tag{9.13}$$

For completeness, we also denote the number change of 4-point vertex with V'.

To make the $\mathcal{O}(e^8)$ out of the $\mathcal{O}(e^6)$, let's explore the $\mathcal{O}(e^6)$ diagrams modified from the diagram I first. Notice the $4 + 4 + 4 + 6 = 18$ diagrams generated by operation $I -$

$(i)/(iii)/(v)/(vii)$, can be further corrected as shown in Eq. (9.14):

$$\begin{aligned}
&I{-}(i)/(iii)/(v)/(vii)\ [S=6,\ P=5,\ V=6,\ V'=0] \text{ followed by } (18\times\cdots): \\
&\quad 5P-(i):\ 5 \quad [S=8,\ P=6,\ V=8,\ V'=0], \\
&\quad 5P-(ii):\ 5 \quad [S=7,\ P=6,\ V=6,\ V'=0], \\
&\quad 6S-(iii)(v):\ 6\times 2=12 \quad [S=8,\ P=6,\ V=8,\ V'=0], \\
&\quad 6S-(iv):\ 6 \quad [S=7,\ P=6,\ V=6,\ V'=1], \\
&\quad 6V-(vi):\ 6 \quad [S=7,\ P=6,\ V=5,\ V'=1], \\
&\quad 6S-(vii):\ C_2^6=\binom{6}{2}=\frac{6!}{4!2!}=15 \quad [S=8,\ P=6,\ V=8,\ V'=0], \\
&\quad 6V-(viii)-6S:\ 6\times 6=36 \quad [S=7,\ P=6,\ V=6,\ V'=1], \\
&\quad 6V-(xi):\ C_2^6=\binom{6}{2}=\frac{6!}{4!2!}=15 \quad [S=6,\ P=6,\ V=4,\ V'=2], \\
&\quad \text{total: } 18\times(5+5+12+6+6+15+36+15)=1800.
\end{aligned} \tag{9.14}$$

Then, the $4+4+16=24$ diagrams generated by operation $I-(ii)/(iv)/(viii)$ can be further corrected as shown in Eq. (9.15):

$$\begin{aligned}
&I{-}(ii)/(iv)/(viii)\ [S=5,\ P=5,\ V=4,\ V'=1] \text{ followed by } (24\times\cdots): \\
&\quad 5P-(i):\ 5 \quad [S=7,\ P=6,\ V=6,\ V'=1], \\
&\quad 5P-(ii):\ 5 \quad [S=6,\ P=6,\ V=4,\ V'=2], \\
&\quad 5S-(iii)(v):\ 5\times 2=10 \quad [S=7,\ P=6,\ V=6,\ V'=1], \\
&\quad 5S-(iv):\ 5 \quad [S=6,\ P=6,\ V=4,\ V'=2], \\
&\quad 4V-(vi):\ 4 \quad [S=6,\ P=6,\ V=3,\ V'=2], \\
&\quad 5S-(vii):\ C_2^5=\binom{5}{2}=\frac{5!}{3!2!}=10 \quad [S=7,\ P=6,\ V=6,\ V'=1], \\
&\quad 4V-(viii)-5S:\ 4\times 5=20 \quad [S=6,\ P=6,\ V=4,\ V'=2], \\
&\quad 4V-(xi):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6 \quad [S=5,\ P=6,\ V=2,\ V'=3], \\
&\quad \text{total: } 24\times(5+5+10+5+4+10+20+6)=1560.
\end{aligned} \tag{9.15}$$

The 4 diagrams generated by operation $I-(vi)$ can be further corrected as shown in Eq.

(9.16):

$$
\begin{aligned}
&I-(vi)\ [S=5,\ P=5,\ V=3,\ V'=1] \text{ followed by } (4\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=7,\ P=6,\ V=5,\ V'=1],\\
&\quad 5P-(ii):\ 5\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 5S-(iii)(v):\ 5\times 2=10\quad [S=7,\ P=6,\ V=5,\ V'=1],\\
&\quad 5S-(iv):\ 5\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 3V-(vi):\ 3\quad [S=6,\ P=6,\ V=2,\ V'=2],\\
&\quad 5S-(vii):\ C_2^5=\binom{5}{2}=\frac{5!}{3!2!}=10\quad [S=7,\ P=6,\ V=5,\ V'=1],\\
&\quad 3V-(viii)-5S:\ 3\times 5=15\quad [S=6,\ P=6,\ V=5,\ V'=2],\\
&\quad 3V-(xi):\ C_2^3=\binom{3}{2}=\frac{3!}{2!1!}=3\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad \text{total: } 4\times(5+5+10+5+3+10+15+3)=224.
\end{aligned}
\tag{9.16}
$$

The 6 diagrams generated by operation $I-(xi)$ can be further corrected as shown in Eq. (9.17):

$$
\begin{aligned}
&I-(xi)\ [S=4,\ P=5,\ V=2,\ V'=2] \text{ followed by } (6\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 5P-(ii):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 4S-(iii)(v):\ 4\times 2=8\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4S-(iv):\ 4\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(vi):\ 2\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 4S-(vii):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 2V-(viii)-4S:\ 2\times 4=8\quad [S=5,\ P=6,\ V=4,\ V'=3],\\
&\quad 2V-(xi):\ C_2^2=1\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad \text{total: } 6\times(5+5+8+4+2+6+8+1)=234.
\end{aligned}
\tag{9.17}
$$

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram I (II) is $1800+1560+224+234=3818$.

Similarly, we can apply the operation on type III and IV diagrams. The $4+3+3+3=13$ diagrams generated by operation $III-(i)/(iii)/(v)/(vii)$, can be further corrected as shown in Eq. (9.18):

$$
\begin{aligned}
&III{-}(i)/(iii)/(v)/(vii)\ [S=5,\ P=5,\ V=4,\ V'=1] \text{ followed by } (13\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 5P-(ii):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=1],\\
&\quad 5S-(iii)(v):\ 5\times 2=10\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 5S-(iv):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4V-(vi):\ 4\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 5S-(vii):\ C_2^5=\binom{5}{2}=\frac{5!}{3!2!}=10\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 4V-(viii)-5S:\ 4\times 5=20\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4V-(xi):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad \text{total: } 13\times(5+5+10+5+4+10+20+6)=845.
\end{aligned}
\tag{9.18}
$$

Then, the $4+3+6=13$ diagrams generated by operation $III-(ii)/(iv)/(viii)$ can be further corrected as shown in Eq. (9.19):

$$
\begin{aligned}
&III{-}(ii)/(iv)/(viii)\ [S=4,\ P=5,\ V=2,\ V'=2] \text{ followed by } (13\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 5P-(ii):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 4S-(iii)(v):\ 4\times 2=8\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4S-(iv):\ 4\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(vi):\ 2\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 4S-(vii):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 2V-(viii)-4S:\ 2\times 4=8\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(xi):\ C_2^2=1\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad \text{total: } 13\times(5+5+8+4+2+6+8+1)=507.
\end{aligned}
\tag{9.19}
$$

The 2 diagrams generated by operation $III-(vi)$ can be further corrected as shown in Eq.

(9.20):

$$
\begin{aligned}
&III\!-\!(vi)\ [S=4,\ P=5,\ V=1,\ V'=2] \text{ followed by } (2\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 5P-(ii):\ 5\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 4S-(iii)(v):\ 4\times 2=8\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 4S-(iv):\ 4\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 1V-(vi):\ 1\quad [S=5,\ P=6,\ V=0,\ V'=3],\\
&\quad 4S-(vii):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 1V-(viii)-4S:\ 1\times 4=4\quad [S=5,\ P=6,\ V=3,\ V'=3],\\
&\quad 1V-(xi):0,\\
&\quad \text{total: } 4\times(5+5+8+4+1+6+4+0)=132.
\end{aligned}
\tag{9.20}
$$

The 1 diagrams generated by operation $III-(xi)$ can be further corrected as shown in Eq. (9.21):

$$
\begin{aligned}
&III\!-\!(xi)\ [S=3,\ P=5,\ V=0,\ V'=3] \text{ followed by } (1\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 5P-(ii):\ 5\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad 3S-(iii)(v):\ 3\times 2=6\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 3S-(iv):\ 3\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad 0V-(vi):\ 0,\\
&\quad 3S-(vii):\ C_2^3=\binom{3}{2}=\frac{3!}{2!1!}=3\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 0V-(viii)-3S:\ 0,\\
&\quad 0V-(xi):\ 0,\\
&\quad \text{total: } 1\times(5+5+6+3+0+3+0+0)=22.
\end{aligned}
\tag{9.21}
$$

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram III (IV) is $845+507+132+22=1506$.

Next, we can apply the operation on type V diagram. The $4+2+2+1=9$ diagrams generated by operation $V-(i)/(iii)/(v)/(vii)$, can be further corrected as shown in Eq.

(9.22):

$$
\begin{aligned}
&V-(i)/(iii)/(v)/(vii)\ [S=4,\ P=5,\ V=2,\ V'=2]\ \text{followed by}\ (9\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 5P-(ii):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=2],\\
&\quad 4S-(iii)(v):\ 4\times 2=8\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4S-(iv):\ 4\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(vi):\ 2\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 4S-(vii):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 2V-(viii)-4S:\ 2\times 4=8\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(xi):\ C_2^2=1\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad \text{total: } 9\times(5+5+8+4+2+6+8+1)=351.
\end{aligned}
\tag{9.22}
$$

Then, the $4+2+0=6$ diagrams generated by operation $V-(ii)/(iv)/(viii)$ can be further corrected as shown in Eq. (9.23):

$$
\begin{aligned}
&V-(ii)/(iv)/(viii)\ [S=3,\ P=5,\ V=0,\ V'=3]\ \text{followed by}\ (6\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 5P-(ii):\ 5\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad 3S-(iii)(v):\ 3\times 2=6\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 3S-(iv):\ 3\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad 0V-(vi):\ 0,\\
&\quad 3S-(vii):\ C_2^3=\binom{3}{2}=\frac{3!}{2!1!}=3\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 0V-(viii)-3S:\ 0,\\
&\quad 0V-(xi):\ 0,\\
&\quad \text{total: } 6\times(5+5+6+3+0+3+0+0)=132.
\end{aligned}
\tag{9.23}
$$

No diagrams can be generated by operation $V-(vi)$.

No diagrams can be generated by operation $V-(xi)$.

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram V is $351+132=483$.

Next, we can apply the operation on type VI diagram. The diagrams generated by adding operation onto VI are listed in Eq. (9.24):

$$\begin{aligned}
&VI\ [S=6,\ P=5,\ V=6,\ V'=0] \text{ followed by } (1\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=8,\ P=6,\ V=8,\ V'=0],\\
&\quad 5P-(ii):\ 5\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 6S-(iii)(v):\ 6\times 2=12\quad [S=8,\ P=6,\ V=8,\ V'=0],\\
&\quad 6S-(iv):\ 6\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 6V-(vi):\ 6\quad [S=7,\ P=6,\ V=5,\ V'=1],\\
&\quad 6S-(vii):\ C_2^6=\binom{6}{2}=\frac{6!}{4!2!}=15\quad [S=8,\ P=6,\ V=8,\ V'=0],\\
&\quad 6V-(viii)-6S:\ 6\times 6=36\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 6V-(xi):\ C_2^6=\binom{6}{2}=\frac{6!}{4!2!}=15\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad \text{total: } 1\times(5+5+12+6+6+16+36+15)=101.
\end{aligned} \tag{9.24}$$

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram VI is 101.

Next, we can apply the operation on type VII diagram. The diagrams generated by adding operation onto VII are listed in Eq. (9.25):

$$\begin{aligned}
&VII\ [S=4,\ P=5,\ V=2,\ V'=2] \text{ followed by } (1\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 5P-(ii):\ 5\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 4S-(iii)(v):\ 4\times 2=8\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4S-(iv):\ 4\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(vi):\ 2\quad [S=5,\ P=6,\ V=1,\ V'=3],\\
&\quad 4S-(vii):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 2V-(viii)-4S:\ 2\times 4=8\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad 2V-(xi):\ C_2^2=1\quad [S=4,\ P=6,\ V=0,\ V'=4],\\
&\quad \text{total: } 1\times(5+5+8+4+2+6+8+1)=39.
\end{aligned} \tag{9.25}$$

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram VII is 39.

Lastly, we can apply the operation on type $VIII$ (IX) diagram. The diagrams generated by adding operation onto them are listed in Eq. (9.26):

$$
\begin{aligned}
&VIII\ [S=5,\ P=5,\ V=4,\ V'=1] \text{ followed by } (1\times\cdots):\\
&\quad 5P-(i):\ 5\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 5P-(ii):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 5S-(iii)(v):\ 5\times 2=10\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 5S-(iv):\ 5\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4V-(vi):\ 4\quad [S=6,\ P=6,\ V=3,\ V'=2],\\
&\quad 5S-(vii):\ C_2^5=\binom{5}{2}=\frac{5!}{3!2!}=10\quad [S=7,\ P=6,\ V=6,\ V'=1],\\
&\quad 4V-(viii)-5S:\ 4\times 5=20\quad [S=6,\ P=6,\ V=4,\ V'=2],\\
&\quad 4V-(xi):\ C_2^4=\binom{4}{2}=\frac{4!}{2!2!}=6\quad [S=5,\ P=6,\ V=2,\ V'=3],\\
&\quad \text{total: } 1\times(5+5+10+5+4+10+20+6)=65.
\end{aligned}
\tag{9.26}
$$

Therefore, the number of $\mathcal{O}(e^8)$ diagrams that can be gotten by correction on diagram $VIII$ (IX) is 65.

At $\mathcal{O}(e^8)$ in total, we have $3818\times 2+1506\times 2+483+101+39+65\times 2=11401$ diagrams. By the way, the diagram shown in the Eq. (9.68) of the book can be gotten by operation (vii) onto the diagram VI.

(b) Gauge invariance means the diagram should be independent of the gauge choice ξ variable in the photon propagator. There are two internal photon propagators in the graph. One can effectively treat this diagram as the t-channel diagrams as Eq. (9.41) of the book. Thus, the diagram is gauge invariant when also include the diagram with the photon lines like a u-channel diagrams and the 4-point vertex like the Eq. (9.43) of the book. The only subtle point however is now one can no longer treat the scalar as on-shell because they are internal lines. To show the gauge-dependent part of the diagrams still cancel out, we can write out the integral like the Eq. (9.44) of the book but don't impose the on-shellness of the scalar particle.

$$
\begin{aligned}
\mathcal{M}_t+\mathcal{M}_u+\mathcal{M}_4 = e^2\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)\frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2}\\
\times\left[\frac{(q^\mu-2p_1^\mu)(k^\nu-2p_2^\nu)}{(q-p_1)^2-m^2}+\frac{(q^\mu-2p_2^\mu)(k^\nu-2p_1^\nu)}{(q-p_2)^2-m^2}+2g^{\mu\nu}\right]\Pi_{\mu\alpha}(q)\Pi_{\nu\beta}(k)X_{\alpha\beta}(q,k).
\end{aligned}
\tag{9.27}
$$

Now, if we replace $\Pi_{\mu\alpha}(q)\to\xi q_\mu q_\alpha$, throwing the parts vanish due to the momentum conservation, we're left with

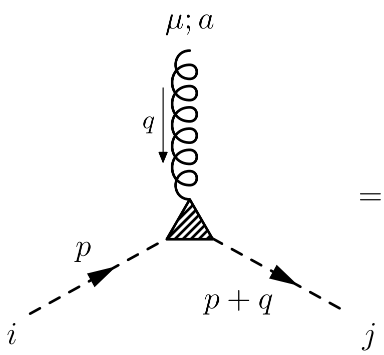
$$\begin{aligned}
\mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_4 \to\; & e^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)\frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2} \\
& \times\left[\frac{(-p_1^2+m^2)(k^\nu-2p_2^\nu)}{(q-p_1)^2-m^2}+\frac{(-p_2^2+m^2)(k^\nu-2p_1^\nu)}{(q-p_2)^2-m^2}\right]\Pi_{\nu\beta}(k)X_{\alpha\beta}(q,k) \\
=\;& e^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\
& \times\left[\frac{(-k^\nu+2p_2^\nu)}{((q-p_1)^2-m^2)(p_2^2-m^2)}+\frac{(-k^\nu+2p_1^\nu)}{((q-p_2)^2-m^2)(p_1^2-m^2)}\right]\Pi_{\nu\beta}(k)X_{\alpha\beta}(q,k) \\
=\;& e^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\
& \times\left[\frac{(-k^\nu+2p_2^\nu-2q^\nu)}{(p_1^2-m^2)((p_2-q)^2-m^2)}+\frac{(-k^\nu+2p_1^\nu)}{((q-p_2)^2-m^2)(p_1^2-m^2)}\right]\Pi_{\nu\beta}(k)X_{\alpha\beta}(q,k) \\
=\;& e^2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)\left[\frac{(-2k^\nu+2p_2^\nu-2q^\nu+2p_1)}{(p_1^2-m^2)((p_2-q)^2-m^2)}\right]\Pi_{\nu\beta}(k)X_{\alpha\beta}(q,k \\
=\;& 0
\end{aligned} \tag{9.28}$$

where to get the third to last line, we simultaneously linear shift the dummy momentum $p_1 \to p_1 + q$ and $p_2 \to p_2 - q$ in the first integral. The next to last line clearly vanishes because of the momentum conservation.

9.3

I shall take polarization vectors to be real to save time writing a * for every external photon polarizations. I would suggest to do Problem 9.4 part (a) first before approaching this one.

(a) By Lorentz invariance,



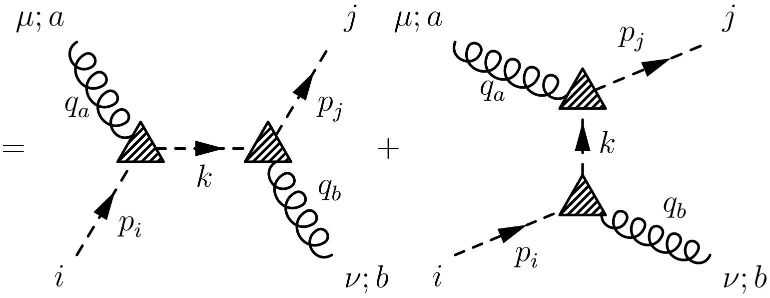
$$= -ig\Gamma^{a\mu}_{ij}(p,q) = -2igp^\mu F^a_{ij}(p^2,q^2,p\cdot q) = -2igp^\mu F^a_{ij}\left(\frac{p\cdot q}{m_i^2}\right), \tag{9.29}$$

where we discard the form factor G^a_{ij} related to q^μ because for external photon, $q\cdot\epsilon = 0$, and put on-shell condition in the last step.

In the soft limit, $q \ll p$, the form factor just go to $F^a_{ij}\left(\frac{p\cdot q}{m_i^2}\right) \to F^a_{ij}(0) \equiv T^a_{ij}$, which is just a constant matrix.

(b) Consider Compton scattering, $\phi_i(p_i)A^a_\mu(q_a) \to \phi_j(p_j)A^b_\nu(q_b)$ in the soft limit $q^a, q^b \ll p_i, p_j$. With only 3-point vertex, there are two diagrams could contribute:

$$i\mathcal{M} = i\mathcal{M}_s + i\mathcal{M}_t$$



$$= i(-2ig)^2\epsilon^\mu_a \sum_k \left[\frac{T^a_{ik}T^b_{kj}p^\mu_i(p_j\cdot\epsilon_b)}{m_i^2 - m_k^2 + 2p_i\cdot q_a} + \frac{T^b_{ik}T^a_{kj}(p_i\cdot\epsilon_b)p^\mu_j}{m_j^2 - m_k^2 - 2p_j\cdot q_a}\right]$$
$$\equiv i\epsilon^\mu_a M_\mu, \tag{9.30}$$

where the sum is over all scalar fields. By consistency condition of little group transformation, it must be $q^\mu_a M_\mu = 0$. For $N = 1$, this cannot happen for $m_i \neq m_k$ or $m_i \neq m_j$ unless $T^{a,b} = 0$. Hence, we conclude that

$$\boxed{[M, T] = 0}\,. \tag{9.31}$$

That is the gluons (or the photon) can only couple between particles of the same mass.

(c)

$$\begin{aligned} q^\mu_a M_\mu &= -2g^2\left[(T^aT^b)_{ij}(p_j\cdot\epsilon_b) - (T^bT^a)_{ij}(p_i\cdot\epsilon_b)\right] \\ &= -g^2\left[\left(\left\{T^a,T^b\right\} + \left[T^a,T^b\right]\right)_{ij}(p_j\cdot\epsilon_b) - \left(\left\{T^a,T^b\right\} - \left[T^a,T^b\right]\right)_{ij}(p_i\cdot\epsilon_b)\right] \\ &= -g^2\left[((p_j - p_i)\cdot\epsilon_b)\left\{T^a,T^b\right\}_{ij} + ((p_j + p_i)\cdot\epsilon_b)\left[T^a,T^b\right]_{ij}\right] \\ &= -g^2\epsilon^\nu_b\left[q^\nu_a\left\{T^a,T^b\right\}_{ij} + (2p_i + q_a)^\nu\left[T^a,T^b\right]_{ij}\right] \end{aligned} \tag{9.32}$$

Note that the sum on k has been recasted into matrix product of T. I also used momentum conservation $p_i + q_a = p_j + q_b$ and on-shellness of external massless spin-1 particles $q_a^2 = \epsilon_b\cdot q_b = 0$. **This does not vanish in general even if one sets $\left[T^a, T^b\right] = 0$ since there is also a piece $\propto \left\{T^a, T^b\right\}$. Both terms need to introduce quartic interactions $\phi_i\phi_j A^a_\mu A^b_\nu$ to cancel with such that Ward identity is satisfied. In fact, the piece $\propto \left\{T^a, T^b\right\}$ would be canceled by the local quartic interaction (cf. Problem 9.4 part (a)) while the piece $\propto \left[T^a, T^b\right]$ would require non-local quartic interaction which signals exchange diagram.**

Do note that Ward identity should be satisfied even if one does not take the soft limits kinematics and if one only replace either one of the polarization vector by its momentum.

I shall suppress the indices ij on the form factors $T^{a,b}$ from here on.

Before proceeding, I shall list some useful identities when taking on-shell condition for external states.

$$s = (p_i + q_a)^2 = (p_j + q_b)^2 \implies p_i \cdot q_a = p_j \cdot q_b, \tag{9.33}$$
$$t = (q_a - p_j)^2 = (q_b - p_i)^2 \implies p_j \cdot q_a = p_i \cdot q_b, \tag{9.34}$$
$$u = (q_a - q_b)^2 = (p_j - p_i)^2 \implies q_a \cdot q_b = p_i \cdot p_j - m^2. \tag{9.35}$$

(d) The generic form of the quartic interaction $iA\Gamma_{ij}^{ab\mu\nu}$ is already given by Eq. (9.49) except we now need to add the color indices:

$$\Gamma_{ij}^{ab\mu\nu} = \sum_{k_n, l_n \in \{p_{i,j}, q_{a,b}\}} \frac{1}{(k_n \cdot l_n)} \Big(p_i^\mu p_i^\nu F_n^{ab}(0) + q_b^\mu q_a^\nu G_n^{ab}(0) + q_b^\mu p_i^\nu G_n'^{ab}(0) + p_i^\mu q_a^\nu G_n''^{ab}(0) \Big) + g^{\mu\nu} F_{loc}^{ab}(0). \tag{9.36}$$

Contracting with external polarization vectors $\epsilon_{a,b}$ and replacing $\epsilon_a \to q_a$,

$$A\epsilon_b^\nu q_a^\mu \Gamma_{ij}^{ab\mu\nu} = A\epsilon_b^\nu \Bigg[\sum_{k_n, l_n \in \{p_{i,j}, q_{a,b}\}} \frac{1}{(k_n \cdot l_n)} \big((p_i \cdot q_a) p_i^\nu F_n^{ab}(0) + (q_a \cdot q_b) q_a^\nu G_n^{ab}(0) + (q_a \cdot q_b) p_i^\nu G_n'^{ab}(0) + (p_i \cdot q_a) q_a^\nu G_n''^{ab}(0) \big) + q_a^\nu F_{loc}^{ab}(0) \Bigg] \tag{9.37}$$

To cancel the piece $\propto \Big\{T^a, T^b\Big\}$ in Eq. (9.32) and note that if the interaction is truly local, since the spin-1 particle follow Bose-Einstein statistics, they must commute when exchanging them, we must choose

$$A = g^2, \qquad F_{loc}^{ab}(0) = \Big\{T^a, T^b\Big\}. \tag{9.38}$$

Hence, the Feynman rule for a local quartic interaction is uniquely fixed as

$$\boxed{i\Gamma_4^{ab\mu\nu} = ig^2 \Big\{T^a, T^b\Big\} g^{\mu\nu}}\ . \tag{9.39}$$

Note that when there is only a single massless spin-1 particle such that $T^{a,b} \to F(0)$ and $\Big[T^a, T^b\Big] = 0$, this reduces correctly to the Feynman rule of the scalar QED: $i\Gamma_4^{\mu\nu} = 2ig^2 F(0)^2 g^{\mu\nu}$, and this term alone is enough to preserve Ward identity.

However, as we have seen, there are also terms $\propto \Big[T^a, T^b\Big]$ in Eq. (9.32), which can only be canceled by some non-local terms in Eq. (9.36). For example, one could choose

$$\boxed{\frac{q_b^\mu q_a^\nu}{q_a \cdot q_b} G_{q_a \cdot q_b}^{ab}(0), \qquad \frac{q_b^\mu p_i^\nu}{q_a \cdot q_b} G_{q_a \cdot q_b}'^{ab}(0)}$$

properly with

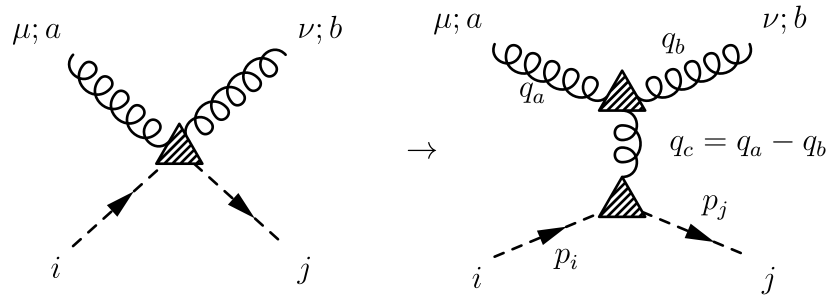
$$G^{ab}_{q_a \cdot q_b}(0), G'^{ab}_{q_a \cdot q_b}(0) \propto \Big[T^a, T^b\Big].$$

As can be seen from Eq. (9.37), upon contracting with $\epsilon_a \to q_a$, these become

$$\propto q_a^{\nu}\Big[T^a, T^b\Big], \qquad \propto p_i^{\nu}\Big[T^a, T^b\Big]$$

which is the right form to cancel the terms $\propto \Big[T^a, T^b\Big]$ in Eq. (9.32) such that Ward identity is preserved. These non-local quartic vertex have poles at $q_a \cdot q_b = (q_a - q_b)^2 \to 0$ as expected. One can replace instead $\epsilon_b \to q_b$ or both and one would still find one needs non-local vertex that have poles at $q_a \cdot q_b = (q_a - q_b)^2 \to 0$. An intuitive way to understand why this always happen and why the poles are always at $(q_a - q_b)^2 \to 0$ is to note that for Ward identity to be preserved, the leftover uncanceled poles in s- and t- channels can only be canceled with a pole in a fictitious u-channel. However such a fictitious u-channel cannot exist unless the multiple massless spin-1 particles have self-interaction. In conclusion, these non-local vertex must be resolved into a local exchange diagram, which generates the pole from propagator.

(e) Resolving the non-local vertex into an exchange diagram means



(9.40)

$$\Rightarrow i\mathcal{M}_u \sim (ig)^2 \frac{\Gamma^{abc}_{\mu\nu\alpha}(q^a, q^b, q^c) \times \Gamma^{c\alpha}_{ij}(p_i, q^c)}{(q_a - q_b)^2} \epsilon^{\mu}_a \epsilon^{\nu}_b.$$

The soft limit in this case corresponds to the internal photon $q_c^2 = (q_a - q_b)^2 \to 0$ goes on shell. Hence, we can again use the form

$$\Gamma^{c\alpha}_{ij}(p_i, q^c) \to 2p_i^{\alpha} T^c, \tag{9.41}$$

which is supposed to be valid only when attached an on-shell massless spin-1 particle in the soft limit. Also note that the coupling strength of the trilinear self-interaction must also be $\propto g$.

By Lorentz invariance, and recall we assume $\Gamma^{abc}_{\mu\nu\alpha}$ has no pole:

$$\Gamma^{abc}_{\mu\nu\alpha}(q^a, -q^b, -q^c) = f^{abc}[g^{\mu\nu}(q_a - q_b)^{\alpha} + g^{\nu\alpha}(q_b - q_c)^{\mu} + g^{\alpha\mu}(q_c - q_a)^{\nu}], \tag{9.42}$$

where f^{abc} is some constant carrying color indices. Note we used a convention that $q^a, -q^b, -q^c$ are incoming to the vertex. Due to momentum conservation, any other momentum dependence either vanishes when contracting with external polarization vector or can be expressed into above. Also, terms in the square bracket is the unique one that is anti-symmetric upon simultaneously permuting any two color indices and their corresponding

Lorentz indices, which thus forces f^{abc} to be an operator that must also be anti-symmetric under permuting any two color indices. This is the desired property if one wants to cancel the pieces $\propto \left[T^a, T^b\right]$ in Eq. (9.32).

We can now check explicitly these are exactly what we needed to preserve Ward identity. Let $\mathcal{M}_u \equiv \epsilon_a^\mu M_{u,\mu}$. Then,

$$\begin{aligned}
q_a^\mu M_{u,\mu} &= -ig^2\epsilon_b^\nu \frac{q_a^\mu p_i^\alpha T^c \Gamma^{abc}_{\mu\nu\alpha}(q^a, q^b, q^c)}{q_a \cdot q_b} \\
&= -ig^2\epsilon_b^\nu \frac{f^{abc}T^c\left[p_i \cdot (q_a + q_b)q_a^\nu + q_a \cdot (-q_b + q_c)p_i^\nu - (p_i \cdot q_a)(q_c + q_a)^\nu\right]}{q_a \cdot q_b} \\
&= -ig^2\epsilon_b^\nu \frac{f^{abc}T^c\left[(p_i \cdot q_b)q_a^\nu - 2(q_a \cdot q_b)p_i^\nu - (p_i \cdot q_a)q_a^\nu\right]}{q_a \cdot q_b} \\
&= -ig^2\epsilon_b^\nu \frac{f^{abc}T^c\left[((p_j + q_b - q_a) \cdot q_b)q_a^\nu - 2(q_a \cdot q_b)p_i^\nu - (p_i \cdot q_a)q_a^\nu\right]}{q_a \cdot q_b} \\
&= ig^2\epsilon_b^\nu f^{abc}T^c\left(q_a^\nu + 2p_i^\nu\right),
\end{aligned} \tag{9.43}$$

where we used on-shellness $q_{a,b}^2 = \epsilon_b \cdot q_b = 0$ multiple times, momentum conservation, as well as $p_j \cdot q_b = p_i \cdot q_a$ from Eq. (9.33).

Combining this, the local quartic contribution Eq. (9.37), and the s-, t-channel contributions to Compton scattering $\phi_i(p_i)A_\mu^a(q_a) \to \phi_j(p_j)A_\nu^b(q_b)$ in Eq. (9.32), we found

$$q_a^\mu M_\mu = q_a^\mu(M_{s,\mu} + M_{t,\mu} + M_{u,\mu} + M_{4,\mu}) = g^2[\epsilon_b \cdot (2p_i + q_a)]\left(if^{abc}T^c - \left[T^a, T^b\right]\right) \tag{9.44}$$

This must vanish to be consistent with the little group transformation such that Ward identity to be satisfied. Since this holds for any p_i and q_a, we conclude that

$$\boxed{\left[T^a, T^b\right] = if^{abc}T^c}\,. \tag{9.45}$$

That is the gluons must transform in the adjoint representation of a Lie group.

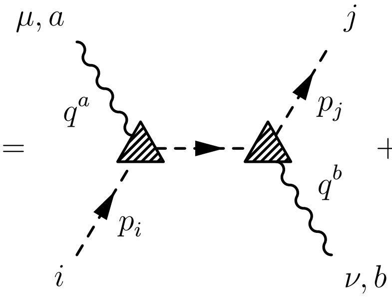
9.4

I shall take polarization vectors/tensors to be real to save time writing a * for every external photon/graviton polarizations.

(a) Consider Compton scattering, $\phi_i(p_i)A_\mu^a(q_a) \to \phi_j(p_j)A_\nu^b(q_b)$ in the soft limit $q_a, q_b \ll p_i, p_j$. With only generic 3-point vertex (cf. Eq. (9.57) of the textbook), there are two diagrams could contribute:

$$i\mathcal{M} = i\mathcal{M}_s + i\mathcal{M}_t$$

$=$



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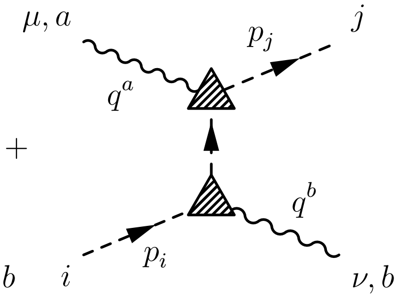


$$= i(-2igF(0))^2 \left[\frac{(p_i \cdot \epsilon_a)(p_j \cdot \epsilon_b)}{2p_i \cdot q_a} + \frac{(p_i \cdot \epsilon_b)(p_j \cdot \epsilon_a)}{-2p_j \cdot q_a} \right]$$
$$= -2ig^2 F(0)^2 \epsilon_a^\mu \left[\frac{p_i^\mu (p_j \cdot \epsilon_b)}{p_i \cdot q_a} - \frac{(p_i \cdot \epsilon_b) p_j^\mu}{p_j \cdot q_a} \right]$$
$$\equiv i\epsilon_a^\mu M_\mu. \tag{9.46}$$

Ward identity (cf. Eq. (8.91) of the textbook) requires

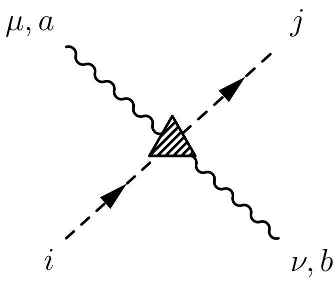
$$q_a^\mu M_\mu = 0, \tag{9.47}$$

which is clearly violated because

$$q_a^\mu M_\mu = -2g^2 F(0)^2 \epsilon_b \cdot (p_j - p_i) = -2g^2 F(0)^2 \epsilon_b \cdot q_a \neq 0 \tag{9.48}$$

in general and note that the non-vanishing part $\propto q_a^\mu$. Hence, we need to add other Lorentz-invariant interactions. In 4D spacetime, there is one more renormalizable interaction, the quartic interaction $AA\phi^*\phi$, we can add.

By Lorentz invariance, a generic quartic contact interaction should take the form



$$= \quad iA\Gamma_4^{\mu\nu}(p_i, q_a, q_b)$$

$$\xrightarrow{\text{soft limit}} iA \left[\sum_{k_n, l_n \in \{p_{i,j}, q_{a,b}\}} \frac{1}{(k_n \cdot l_n)} \big(p_i^\mu p_i^\nu F_n(0) + q_b^\mu q_a^\nu G_n(0) + q_b^\mu p_i^\nu G_n'(0) + p_i^\mu q_a^\nu G_n''(0) \big) + g^{\mu\nu} F_{loc}(0) \right], \tag{9.49}$$

where A is a dimensionless constant. The momentum products in the denominators are inserted to ensure the dimensional correctness since one can embed such a quartic vertex into a 2 $\to$ 2 scattering process of which in 4D spacetime, the amplitude should be dimensionless. Note $k_n \neq l_n$.[12] Also, due to momentum conservation, we can reabsorbed terms $\propto p^j_{\mu,\nu}$ into others and ignore terms that vanish upon contracting with external polarization vector. The A is some arbitrary dimensionless coefficients that are independent of momentum. In the soft limit, again, only $F(0)$ matters. One can think that the argument of F is some linear combinations of $k_m \cdot l_m$. The dimensionlessness of the form factor requires it to be only $F\left(\frac{\sum c_m(k_m \cdot l_m)}{\sum c_n(k_n \cdot l_n)}\right)$. However, since form factor must be analytic in soft limit and one can also use momentum conservation to see the non-vanishing part of the denominator in soft limit can only be m^2. Hence, only $F(0)$ matters in the soft limit[3]. We again discard the form factor that are associated with q_a or q_b and taking the soft limit.

$$i\mathcal{M}_4 = iA\epsilon^\mu_a \epsilon^\nu_b \Gamma_{4,\mu\nu}. \tag{9.50}$$

To restore the Ward identity from Eq. (9.48), we need to choose the form factors in Eq. (9.49) properly such that

$$A\epsilon^\nu_b q^\mu_a \Gamma_{4,\mu\nu} = 2g^2 F(0)^2 \epsilon^\nu_b \cdot q^\nu_a \tag{9.51}$$

It's clear that we need a piece $\propto q^\mu_a$ in M'^μ to cancel that in Eq. (9.48). In this case, just the local term is enough to preserve Ward identity and discard $F_n(0)$, $G_n(0)$, $G'_n(0)$ $G''_n(0)$.... Since A is momentum-independent constant, we must have $A \propto g^2$, and we can simply set

$$A = 2g^2, \qquad F_{loc}(0) = F(0)^2 \tag{9.52}$$

Then,

$$i\mathcal{M} = i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_4 \to 2ig^2 F(0)^2 \epsilon_b \cdot (-q_a + q_a) = 0, \tag{9.53}$$

as expected. Thus, we conclude, for Ward identity to be satisfied, there must be a quartic interaction among scalars and the massless spin-1 particles as

$$\boxed{iA\Gamma_{\mu\nu} = 2ig^2 g_{\mu\nu} F(0)^2}\,, \tag{9.54}$$

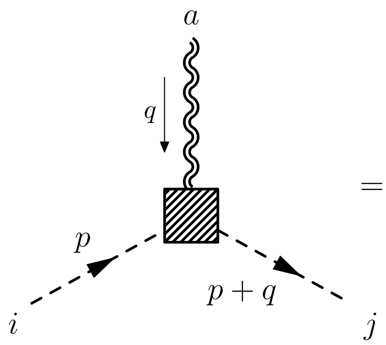
which uniquely fixes the quartic interaction.

[1] BTW, these denominators indicate the vertex can be non-analytic when taking soft limit, if the form factors are non-zero, would indicate there are non-local interactions, which signal the massless spin-1 particle must have self-interaction.

[2] You might also wonder why the poles are in the form of $1/(k_n \cdot l_n)$ instead of a more general linear combinations form $1/(\sum c_n(k_n \cdot l_n))$. Here is the explanation. As a quadratic form, the denominator can have at most one pole. In the soft limit, replacing one of the external polarization vector by it momentum should results in the numerator to have a factor of either $(p_m \cdot q_n)$ or $(q_m \cdot q_n)$. Then, if the soft limit hits the pole in the denominator while the numerator does not vanish, this results in an ill-defined non-analytic contribution and cannot be canceled. Hence, one must set form factor of such terms to be 0. On the other hand, if the numerator goes to 0 in the soft limit while the denominator does not, the contribution vanishes and can be dropped. A third possibility is if neither numerator nor denominator vanishes, in which case results in a constant contribution, which again has no other terms can cancel it and their form factors must also be set to 0. The only valid possibility is both numerator and denominator goes to 0 together in the soft limits.

[3] You might wonder what if one has $F\left(\frac{c_m m^2 + \cdots}{c_n m^2}\right) \to F(c_m/c_n)$ in the soft limit. The answer is you can. The key message however is the form factor F should reduce to a constant with no kinematics dependence in the soft limit, whether you wrote it as a $F(0)$ or $F(c_m/c_n)$ does not really matter.

(b) By Lorentz invariance and dimension analysis[4],



$$= -\frac{i}{M_{\rm Pl}}\Gamma^{\mu\nu}(p,q) = -\frac{2i}{M_{\rm Pl}}p^\mu p^\nu \tilde{F}(0), \tag{9.55}$$

where we discard the form factor G related to q^μ, and put on-shell condition in the last step. Also note that since graviton is a massless spin-2 particle, its polarization tensor $\epsilon^{\mu\nu}$ is traceless and symmetric. Hence, a term associated with $g^{\mu\nu}$ also vanish when contracting with an external graviton and thus we also ignored.

Then, consider Compton scattering in the soft limit with only 3-point vertex.

$$\begin{aligned} i\mathcal{M} &= i\mathcal{M}_s + i\mathcal{M}_t \\ &= \frac{-2i\tilde{F}(0)^2}{M_{\rm Pl}^2}\left[\frac{(p_i^\mu p_i^\nu \epsilon^a_{\mu\nu})(p_j^\alpha p_j^\beta \epsilon^b_{\alpha\beta})}{p_i\cdot q_a} - \frac{(p_i^\alpha p_i^\beta \epsilon^b_{\alpha\beta})(p_j^\mu p_j^\nu \epsilon^a_{\mu\nu})}{p_j\cdot q_a}\right] \\ &\equiv i\epsilon^a_{\mu\nu}M^{\mu\nu} \end{aligned} \tag{9.56}$$

Again, Ward identity says

$$\left[q^a_\mu \Lambda^a_\nu(q^a) + \Lambda^a_\mu(q^a) q^a_\nu\right] M^{\mu\nu} = 0, \tag{9.57}$$

where Λ^ν_a is some Lorentz vector. But this clearly does not vanish:

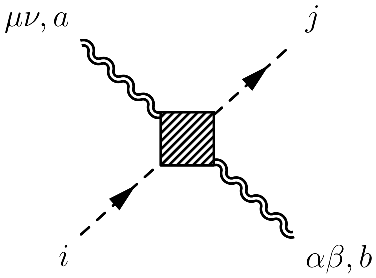
$$\left[q^a_\mu \Lambda^a_\nu(q^a) + \Lambda^a_\mu(q^a) q^a_\nu\right] M^{\mu\nu} = \frac{-2\tilde{F}(0)^2}{M_{\rm Pl}^2}\epsilon^b_{\alpha\beta}\left[\Lambda^a_\mu\left(p_i^\mu p_j^\alpha p_j^\beta - p_j^\mu p_i^\alpha p_i^\beta\right) + \left(p_i^\nu p_j^\alpha p_j^\beta - p_j^\nu p_i^\alpha p_i^\beta\right)\Lambda^a_\nu\right] \neq 0. \tag{9.58}$$

Thus, there must be a quartic interaction $hh\phi\phi$.

Again, the symmetric, traceless, and transverse property of an on-shell massless spin-2 particles constrained the possible form of the general non-vanishing 4-point interaction when attached to an external graviton in the soft limit. Let the two external gravitons carry polarization tensors $\epsilon^{\mu\nu}$ and $\epsilon^{\alpha\beta}$, respectively. The non-vanishing vertex indices must be symmetric under $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. There cannot be any factors of $g^{\mu\nu}$ and $g^{\alpha\beta}$. Also, momentum conservation enforces this can depend on at most 3 external momentum.

[4]Note this is different from the textbook. I don't think the Eq. (9.61) of the textbook is dimensionally correct. I also believe what Weinberg proves for the universality for massless spin-2 is actually the ratio of gravitational mass to inertial mass, not the $M_{\rm Pl}$.

Using the similar procedure as in part (a), the quartic coupling should take the forms as



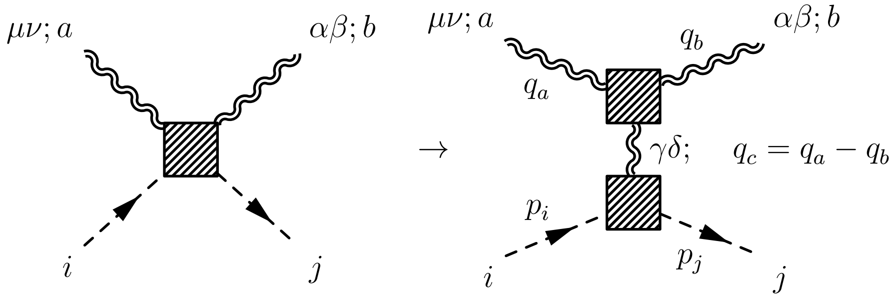
$$= \frac{i}{M_{\mathrm{Pl}}^2}\Gamma^{\mu\nu\alpha\beta}(p_i, q_a, q_b)$$

$$\xrightarrow{\text{soft limit}} \frac{i}{M_{\mathrm{Pl}}^2}\Bigg[\sum_{k,k',l,l'\in\{p_{i,j},q_{a,b}\}} \frac{1}{(k_n\cdot l_n)}\left(k_n^\mu k_n'^\nu l_n^\alpha l_n'^\beta \tilde{F}_n(0)+\cdots\right) + \sum_{k,l\in\{p_{i,j},q_{a,b}\}}\left(k^\mu l^\alpha g^{\nu\beta}\tilde{F}'_n(0)+\cdots\right)+\left(g_{\mu\alpha}g_{\nu\beta}F''_n(0)+\cdots\right)\Bigg]. \tag{9.59}$$

The coupling strength of quartic vertex should be $\sim 1/M_{\mathrm{Pl}}^2$ is clear or this diagram cannot cancel with the other Compton scattering diagrams. Again, the factors $k_n \cdot l_n$ in the first line must be inserted to ensure dimensional consistency, and it's these terms containing poles.

Much like the situation in Problem 9.3, the terms without pole need a local seagull quartic diagram to cancel with while the terms with pole ($\sim \frac{1}{(k_n\cdot l_n)}$) indicates graviton exchange. The same argument as we made in the previous problem ensures there must be a pole at $(q_a+q_b)^2=0$ (a fictions u-channel) to cancel with the poles in s- and t-channels.

Resolving the non-local vertex into an exchange diagram means



(9.60)

$$\Rightarrow i\mathcal{M}_u \sim \left(\frac{i}{M_{\mathrm{Pl}}}\right)^2 \frac{\Gamma^{\gamma\delta}_{\mu\nu;\alpha\beta}(q^a,q^b,q^c)\times\Gamma_{\gamma\delta}(p_i,q^c)}{(q_a-q_b)^2}\epsilon_a^{\mu\nu}\epsilon_b^{\alpha\beta}.$$

The soft limit in this case corresponds to the internal photon $q_c^2=(q_a-q_b)^2=2q_a\cdot q_b\to 0$ goes on shell. Hence, we can again use the form

$$\Gamma^{\mu\nu}(p_i,q^c)\to 2p^\mu p^\nu\tilde{F}(0), \tag{9.61}$$

which is supposed to be valid only when attached an on-shell massless spin-1 particle in the soft limit. Also note that the form factors of the graviton trilinear self-interactions must also be $\tilde{F}(0)$.

By Lorentz invariance, and recall we assume $\Gamma^{abc}_{\mu\nu\alpha}$ has no pole, we thus conclude that $\Gamma^{\gamma\delta}_{\mu\nu;\alpha\beta}(q^a,q^b,q^c)$ should all consistent combinations of **two factors of 4-momentum and**

two or more factors of metric tensors, with a coupling "charge" $\tilde{F}(0)$ and strength $\frac{1}{M_{\text{Pl}}}$. The momentum-space Feynman rule is far from illuminating. However, in the **position space** $p_\mu \to i\partial_\mu$, it's clear the consistent forms satisfying the above criteria are products of two Christoffel symbols. In fact, with two partial derivatives there are only two kinds of combinations one can write down:

$$\boxed{\sim \tilde{F}(0)\left[\left(\frac{1}{2}hg^{\mu\nu}-h^{\mu\nu}\right)\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\nu\alpha}+g^{\mu\nu}\left(\Gamma^{(1)\alpha}_{\mu\beta}\Gamma^{(2)\beta}_{\nu\alpha}+\Gamma^{(2)\alpha}_{\mu\beta}\Gamma^{(1)\beta}_{\nu\alpha}-\Gamma^{(1)\alpha}_{\mu\nu}\Gamma^{(2)\beta}_{\alpha\beta}-\Gamma^{(2)\alpha}_{\mu\nu}\Gamma^{(1)\beta}_{\alpha\beta}\right)\right]} \tag{9.62}$$

where $\Gamma^{(1)\alpha}_{\mu\beta}$ and $\Gamma^{(2)\alpha}_{\mu\beta}$ are the first and second kind of Christoffel symbols, respectively:

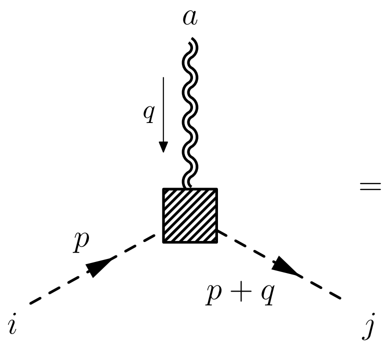
$$\Gamma^{(1)\alpha}_{\mu\nu} = \frac{1}{2}(\partial_\mu h^\alpha_\nu + \partial_\nu h^\alpha_\mu - \partial^\alpha h_{\mu\nu}), \tag{9.63}$$

$$\Gamma^{(2)\alpha}_{\mu\nu} = -\frac{1}{2}h^{\alpha\beta}(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}). \tag{9.64}$$

Side notes for the Lorentz invariance for spin 2

From what I understand from Weinberg's original paper [4], I believe the textbook made a mistake on the interpretation of the universality. The universal "gravitational charge" is certainly not referring to the gravitation strength $\frac{1}{M_{\text{Pl}}} = \sqrt{G_n}$ because all the other gauge interactions are interacting with their respective same strength as well, e.g. QED with e. What Weinberg is proving to be universal is actually the ratio of gravitational mass and inertial mass – equivalence principle.

As already suggesting in the previous problem, the correct form of the scalar graviton cubic interaction should be



$$= -\frac{i}{M_{\text{Pl}}}\Gamma^{\mu\nu}(p,q) = -\frac{2i}{M_{\text{Pl}}}p^\mu p^\nu \tilde{F}(0) \tag{9.65}$$

such that the gravitational form factor $\tilde{F}(0)$ is dimensionless.

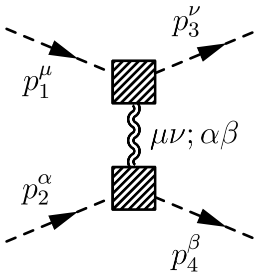
To see the physical meaning of the form factor $\tilde{F}(0)$, we follow Weinberg's approach to note that the classical gravitational potential can be extracted from a t-channel scattering process in the non-relativistic limit ($t \simeq -|\vec{q}|^2$) $\phi_1\phi_2 \xrightarrow{g} \phi_1\phi_2$, which is nothing but just the Born approximation. The graviton propagator can be gotten by inverted the Eq. (8.129) of the textbook[5]:

$$= i\Pi^{\mu\nu;\alpha\beta}(q) = \frac{\frac{i}{2}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)}{q^2}. \tag{9.66}$$

[5]This is of course gauge-dependent. I implicitly use the so called harmonic gauge. The physical amplitude is gauge-independent.

Then, the t-channel gravitational scattering is

$$i\mathcal{M}_t =$$



$$\sim \tilde{F}_1(0)\tilde{F}_2(0)\frac{(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_3\cdot p_2)-(p_1\cdot p_3)(p_2\cdot p_4)}{M_{\text{Pl}}^2 t}$$

$$\begin{aligned}
&= \tilde{F}_1(0)\tilde{F}_2(0)\frac{(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_3\cdot p_2)-(p_1\cdot p_3)(p_2\cdot p_4)}{M_{\text{Pl}}^2 t} \\
&= \tilde{F}_1(0)\tilde{F}_2(0)\frac{2p_{12}^2-m_1^2m_2^2}{M_{\text{Pl}}^2 t},
\end{aligned} \tag{9.67}$$

where I used the following

$$0=t=(p_1-p_3)^2=(p_2-p_4)^2=2m_1^2-2p_{13}=2m_2^2-2p_{24}\Rightarrow\ p_{13}=m_1^2,\ p_{24}=m_2^2 \tag{9.68}$$

$$s=(p_1+p_2)^2=(p_3+p_4)^2=m_1^2+m_2^2-2p_{12}=m_1^2+m_2^2-2p_{34}\Rightarrow p_{12}=p_{34} \tag{9.69}$$

$$u=(p_1-p_4)^2=(p_2-p_3)^2\Rightarrow p_{14}=p_{23} \tag{9.70}$$

$$s+t+u=2m_1^2+2m_2^2\Rightarrow p_{12}=p_{14}=p_{34}=p_{23}. \tag{9.71}$$

Now taking the p_2 to be at rest $p_2=(m_2,0)$ sourcing the potential felt by p_1. Then,

$$i\mathcal{M}_t\sim\tilde{F}_1(0)\tilde{F}_2(0)\frac{2E_1^2m_2^2-m_1^2m_2^2}{M_{\text{Pl}}^2 t} \tag{9.72}$$

The classical potential is gotten from diving the external normalization factors $4E_1E_2$ (recall the normalization convention $\langle\mathbf{p}'|\mathbf{p}\rangle=2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$):

$$\begin{aligned}
\tilde{V}(\mathbf{q}) &= -\frac{\mathcal{M}_t(\mathbf{q})}{4E_1E_2} \\
&\sim \frac{\tilde{F}_1(0)\tilde{F}_2(0)}{4E_1m_2}\frac{2E_1^2m_2^2-m_1^2m_2^2}{M_{\text{Pl}}^2 t} \\
&= -G_n\frac{\left[\tilde{F}_1(0)\left(2E_1-\frac{m_1^2}{E_1}\right)\right]\left[\tilde{F}_2(0)m_2\right]}{\mathbf{q^2}}
\end{aligned} \tag{9.73}$$

A Fourier transform takes the $\frac{1}{\mathbf{q}^2}\to\frac{1}{4\pi r}$. We can match the

$$V(r)=-G_n\frac{\left[\tilde{F}_1(0)\left(2E_1-\frac{m_1^2}{E_1}\right)\right]\left[\tilde{F}_2(0)m_2\right]}{4\pi r} \tag{9.74}$$

with the Newtonian gravitational energy and conclude that the *gravitational mass* (i.e gravitational charge) is

$$m_{grav}\equiv\tilde{F}(0)\left(2E-\frac{m^2}{E}\right) \tag{9.75}$$

of a particle. Now to recognize the physical meaning of the bracket, we can first take the non-relativistic limit of the test particle p_1 and found

$$m_{grav} \equiv \tilde{F}(0)\left(2E - \frac{m^2}{E}\right) \to \tilde{F}(0)(2m - m) = \tilde{F}(0)m \equiv \tilde{F}(0)m_{inert} \;, \tag{9.76}$$

which we recognize it's the mass contributing to the energy-momentum tensor, i.e the *inertial mass.*

Hence,

$$\boxed{\tilde{F}(0) \equiv \frac{m_{grav}}{m_{inert}}} \tag{9.77}$$

in the non-relativistic limit. **That using Weinberg's soft limit of massless spin-1 particle argument, one proves the $\tilde{F}(0)$ must be universal to all particles is hence, proving the equivalence principle**, not the universal strength.

BTW, taking the ultra-relativistic limit of the test particle p_1:

$$m_{grav} \equiv \tilde{F}(0)\left(2E - \frac{m^2}{E}\right) \to 2\tilde{F}(0)E, \tag{9.78}$$

which tells you even massless particles should feel gravitational force.

Chapter 10

Spinors

10.1

(a) Starting from the Dirac equation with covariant derivative,

$$\begin{aligned} i\gamma^\mu(\partial_\mu + ieA_\mu)\psi &= m\psi \\ i\partial_t\psi &= i\gamma^0\gamma^i\partial_i\psi - \gamma^0\gamma^i A_i + m\gamma^0\psi + eA_0\psi \\ &= (\gamma^0\gamma^i(i\partial_i - A_i) + m\gamma^0 + eA_0)\psi, \end{aligned} \tag{10.1}$$

where we multiply γ_0 from left on the second line. We can then identify $H_D = \gamma^0\gamma^i(-p_i - A_i) + m\gamma^0 + eA_0$.

(b) Doing the minimal substitution $p_i \to p_i + eA_i$ in H_D, we have

$$\begin{aligned} (H_D - eA_0)^2 &= (\gamma^0\gamma^i(p_i + eA_i) - m\gamma^0)^2 \\ &= m^2 - m(\gamma^0\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0) + \gamma^0\gamma^i\gamma^0\gamma^j(p_i + eA_i)(p_j + eA_j) \\ &= m^2 - 0 - \gamma^i\gamma^j(p_i + eA_i)(p_j + eA_j) \\ &= m^2 - \frac{1}{2}(\{\gamma^i, \gamma^j\} + [\gamma^i, \gamma^j])(p_i + eA_i)(p_j + eA_j) \\ &= m^2 - \frac{1}{2}(-2\delta^{ij} - 2i\sigma^{ij})(p_i + eA_i)(p_j + eA_j) \\ &= (m^2 + (\vec{p} + e\vec{A})^2)\mathbb{I} + e\vec{B}\cdot\vec{\sigma}, \end{aligned} \tag{10.2}$$

where we used the results of eq.(10.103) - eq.(10.109) from the book in the last line. Put back the factor of c and $\hbar$, we arrive at

$$(H_D - eA_0)^2 = (m^2c^4 + (\vec{p}c + e\vec{A})^2)\mathbb{I} + e\hbar\vec{B}\cdot\vec{\sigma} \tag{10.3}$$

(c) Taking the square root and subtracting off mc^2, we have

$$\begin{aligned} \Big[(m^2c^4 + (\vec{p}c + e\vec{A})^2)\mathbb{I} + e\hbar\vec{B}\cdot\vec{\sigma}\Big]^{\frac{1}{2}} - mc^2 &= mc^2\left[\mathbb{I} + \frac{(\vec{p}c + e\vec{A})^2}{m^2c^4}\mathbb{I} + \frac{e\hbar}{m^2c^4}\vec{B}\cdot\vec{\sigma}\right]^{\frac{1}{2}} - mc^2 \\ &\approx \frac{(\vec{p}c + e\vec{A})^2}{2mc^2}\mathbb{I} + \frac{e\hbar}{2mc^2}\vec{B}\cdot\vec{\sigma} + \dots \end{aligned} \tag{10.4}$$

(d)

$$\left[S_i, S_j\right] = \frac{1}{4}\left[\sigma_i, \sigma_j\right] = \frac{1}{4} \cdot 2i\epsilon_{ijk}\sigma_k = i\epsilon_{ijk}S_k \tag{10.5}$$

$$\begin{aligned}\left[L_i, L_j\right] &= \left[\epsilon_{ikl}x_k p_l, \epsilon_{jmn}x_m p_n\right] = \epsilon_{ikl}\epsilon_{jmn}(x_k p_l x_m p_n - x_m p_n x_k p_l) \\ &= -i\epsilon_{ikl}\epsilon_{jmn}(x_k \delta_{lm} p_n - x_m \delta_{nk} p_l) \\ &= -i(\delta_{kj}\delta_{in} - \delta_{kn}\delta_{ij})x_k p_n + i(\delta_{im}\delta_{lj} - \delta_{ij}\delta_{lm})x_m p_l \\ &= -ix_j p_i + ix \cdot p\delta_{ij} + ix_i p_j - i\delta_{ij} x \cdot p \\ &= i(x_i p_j - x_j p_i)\end{aligned} \tag{10.6}$$

On the other hand,

$$i\epsilon_{ijk}L_k = i\epsilon_{ijk}\epsilon_{klm}x_l p_m = i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})x_l p_m = i(x_i p_j - x_j p_i) \tag{10.7}$$

Thus, $\left[L_i, L_j\right] = i\epsilon_{ijk}L_k$. The angular momentum operator L_i and the spin operator S_i both satisfy the rotation algebra.

(e) Let

$$\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}, \tag{10.8}$$

such that

$$\vec{A} = \frac{1}{2}B_0 \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{10.9}$$

The term that is linear in $\vec{A}$ in Eq.(10.4) is

$$\begin{aligned}\frac{1}{2m}(e\vec{p} \cdot \vec{A} + e\vec{A} \cdot \vec{p}) &= \frac{e}{2m}(p_x A_x + p_y A_y + A_x p_x + A_y p_y) \\ &= \frac{e}{2m}\frac{1}{2}B_0(p_x(-y) + p_y x + (-y)p_x + xp_y) \\ &= \frac{e}{2m}B_0(xp_y - yp_x) \\ &= \frac{e}{2m}B_0 L_z\end{aligned} \tag{10.10}$$

The spin term in Eq.(10.4) is

$$\frac{e}{2m}\vec{B} \cdot \vec{\sigma} = \frac{e}{2m}B_0\sigma^3 \tag{10.11}$$

Thus from the entire coupling term,

$$\frac{e}{2m}B_0(L_z + \sigma^3) = \frac{e}{2m}B_0(L_z + 2S_z), \tag{10.12}$$

we can read off $g_e = 2$.

(f)

(g)

(h)

10.2

(a) First notice

$$\left[\tau_3, \tau^{\pm}\right] = [\tau_3, \tau_1] \pm i[\tau_3, \tau_2] = i\tau_2 \pm \tau_1 = \pm\tau^{\pm}. \tag{10.13}$$

Thus,

$$\tau_3\tau^{\pm}V_j = (\pm\tau^{\pm} + \tau^{\pm}\tau_3)V_j = \tau^{\pm}(\pm 1 + \tau_3)V_j = (\lambda_j \pm 1)\tau^{\pm}V_j. \tag{10.14}$$

However, since τ_3 is an $n \times n$ matrix, it can at most have n eigenstates. But the ladder operators $\tau^{\pm}$ when applying to an eigenstate of τ_3 monotonically increasing(decreasing) the eigenvalue by one unit, there must be at least an eigenstate $V_{max}(V_{min})$ that has the largest(smallest) eigenvalue, which when was acted by $\tau^{\pm}$ should vanish to keep Eq. (10.14) consistent.

(b) The uniqueness of V_{max} comes from the argument in (a) as well as the fact that the representation is irreducible. Irreducibility guarantees that each eigenstate must have distinct eigenvalue.

(c) Notice

$$\left[\tau^+, \tau^-\right] = -i[\tau_1, \tau_2] + i[\tau_2, \tau_1] = 2\tau_3, \tag{10.15}$$

and

$$\left\{\tau^+, \tau^-\right\} = \tau^+\tau^- + \tau^-\tau^+ = 2\tau_1^2 + 2\tau_2^2. \tag{10.16}$$

Also, the Casimir operator is defined to be τ_i^2, where Einstein summation is assumed. We will prove that the Casimir operator commutes with all the generators.

$$\left[\tau_i^2, \tau_j\right] = \tau_i\left[\tau_i, \tau_j\right] + \left[\tau_i, \tau_j\right]\tau_i = i\varepsilon_{ijk}\{\tau_i, \tau_k\} = 0, \tag{10.17}$$

which vanishes because ε_{ijk} is antisymmetric, but the anticommutator is symmetric. We can express the Casimir operator as $\tau_i^2 = \frac{1}{2}\left\{\tau^+, \tau^-\right\} + \tau_3^2$, and this commutes with all the generators as well as $\tau^{\pm}$. Acting the Casimir operator on V_{min}, we shall get

$$\begin{aligned}
\tau_i^2 V_{min} &= (\frac{1}{2}\left\{\tau^+, \tau^-\right\} + \tau_3^2)V_{min} \\
&= (\frac{1}{2}(\tau^+\tau^- + \tau^-\tau^+) + \tau_3^2)V_{min} \\
&= \frac{1}{2}\tau^-\tau^+ + \tau_3^2 V_{min} \\
&= -\frac{1}{2}\left[\tau^+, \tau^-\right] + \tau_3^2 V_{min} \\
&= (-\tau_3 + \tau_3^2)V_{min} \\
&= (j - N)(j - N - 1)V_{min}.
\end{aligned} \tag{10.18}$$

But we could also have

$$\begin{aligned}
\tau_i^2 V_{min} &= \tau_i^2 (\tau^-)^N V_{max} \\
&= (\tau^-)^N \tau_i^2 V_{max} \\
&= (\tau^-)^N (\frac{1}{2}\{\tau^+, \tau^-\} + \tau_3^2) V_{max} \\
&= (\tau^-)^N (\frac{1}{2}(\tau^+\tau^- + \tau^-\tau^+) + \tau_3^2) V_{max} \\
&= (\tau^-)^N (\frac{1}{2}\tau^+\tau^- + \tau_3^2) V_{max} \\
&= (\tau^-)^N (\frac{1}{2}[\tau^+, \tau^-] + \tau_3^2) V_{max} \\
&= (\tau^-)^N (\tau_3 + \tau_3^2) V_{max} \\
&= j(1+j)(\tau^-)^N V_{max} \\
&= j(1+j) V_{min}.
\end{aligned} \tag{10.19}$$

Equating Eq. (10.18) and Eq. (10.19),

$$\begin{aligned}
& j^2 + N^2 - 2Nj - j + N = j^2 + j \\
& N^2 - (2j-1)N - 2j = 0 \\
& N = \frac{(2j-1) \pm \sqrt{(2j-1)^2 + 8j}}{2} \\
& N = (j - \frac{1}{2}) \pm (j + \frac{1}{2}) \\
& N = 2j,
\end{aligned} \tag{10.20}$$

since N must be a non-negative integer, only the plus sign can be chosen. This also tells us j must either be a positive half integer or positive integer.

(d) For $n = 5$, $j = 2$, and $N = 4$. We thus know that in the basis that diagonalizes τ_3, the 5 distinct eigenvalues of τ_3 are $2, 1, 0, -1, -2$. Thus we can write out τ_3 in this basis as

$$\tau_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{10.21}$$

Its ith eigenvector has 1 on the ith entry with all other entries 0. We will explicitly construct the ladder operator's representation now. Since $\tau^+(\tau^-)$ raises(lowers) the 1 on ith entry to i+1th(i-1th) entry, while leaving all other entries 0, it's clear that the $\tau^+(\tau^-)$ will only have non-zero entries right above(below) its diagonal. Therefore, we can write

$$\tau^+ = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.22}$$

Also, by **Schur's lemma**, a group element that commutes with all other group elements in any irreducible representation must be proportional to $\mathbb{I}$. Since the Casimir operator commutes with all the generators, it must be proportional to $\mathbb{I}$. From Eq. (10.19), $\tau_i^2 = 2(1+2)\mathbb{I} = 6\mathbb{I}$. Then,

$$\begin{aligned} \tau^+\tau^- &= \frac{1}{2}\left(\{\tau^+,\tau^-\} + [\tau^+,\tau^-]\right) = \tau_i^2 - \tau_3^2 + \tau_3 \\ &= \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{10.23}$$

Combining this with Eq. (10.22), and $\tau^- = (\tau^+)^\dagger$. We simply have

$$|a|^2 = 4, |b|^2 = 6, |c|^2 = 6, |d|^2 = 4. \tag{10.24}$$

We have freedom to choose these to be real and arrive at

$$\tau^+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tau^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}. \tag{10.25}$$

Then,

$$\tau_1 = \frac{1}{2}(\tau^+ + \tau^-) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{10.26}$$

and

$$\tau_2 = \frac{i}{2}(\tau^- - \tau^+) = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}. \tag{10.27}$$

10.3

(a) First, the RHS is just

$$2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} = 2\delta_{\alpha\dot{\alpha}}\delta_{\beta\dot{\beta}} - 2\delta_{\alpha\dot{\beta}}\delta_{\beta\dot{\alpha}}. \tag{10.28}$$

We also need the identity

$$\mathrm{Tr}\left[\sigma_\mu\sigma_\nu\right] = 2\delta_{\mu\nu}. \tag{10.29}$$

To prove this, we know that since Pauli matrices are all traceless, if either $\mu = 0$ or $\nu = 0$, the only non-vanishing case is if both $\mu = \nu = 0$ and thus, $\mathrm{Tr}[\mathbb{I}] = 2$. Now, for $\mu \neq 0$ and $\nu \neq 0$,

$$\mathrm{Tr}\big[\sigma_i \sigma_j\big] = \frac{1}{2}\,\mathrm{Tr}\big[\{\sigma_i, \sigma_j\} + [\sigma_i, \sigma_j]\big] = \delta_{ij}\,\mathrm{Tr}[\mathbb{I}] = 2\delta_{ij} \tag{10.30}$$

The commutator inside the trace vanishes again because Pauli matrices are traceless. Another way to see is because of the cyclic property of trace, the only non-vanishing part is the symmetric term. Thus, Eq. (10.29) is proved.

Next, notice the Pauli matrices and the identity matrix σ^0 form an orthogonal basis for the Hilbert space of 2×2 complex matrices. Thus, any 2×2 complex matrix M can be expressed as

$$M = \sum_\mu a_\mu \sigma^\mu, \tag{10.31}$$

for some complex constants a_μ. Notice this is a simple sum with no sign difference between the 0-th component and other components. The a_μ can be extracted as

$$\mathrm{Tr}[M\sigma^\nu] = \sum_\mu a_\mu\,\mathrm{Tr}[\sigma^\mu \sigma^\nu] = 2\sum_\mu a_\mu \delta^{\mu\nu} = 2a^\nu, \tag{10.32}$$

where Eq. (10.29) is used, and then

$$a_\mu = \frac{1}{2}\,\mathrm{Tr}\big[M\sigma_\mu\big]. \tag{10.33}$$

So

$$2M = \sum_\mu \mathrm{Tr}[M\sigma^\mu]\sigma^\mu \tag{10.34}$$

or written out with spinor indices,

$$\begin{aligned} 2M_{\alpha\dot{\alpha}} &= \sum_\mu \sigma^\mu_{\alpha\dot{\alpha}} M_{\dot{\beta}\beta} \sigma^\mu_{\beta\dot{\beta}} \\ 0 &= M_{\dot{\beta}\beta}\big(\sum_\mu \sigma^\mu_{\alpha\dot{\alpha}} \sigma^\mu_{\beta\dot{\beta}} - 2\delta_{\beta\dot{\alpha}}\delta_{\dot{\beta}\alpha}\big). \end{aligned} \tag{10.35}$$

Since M is an arbitrary matrix, we must have

$$\sum_\mu \sigma^\mu_{\alpha\dot{\alpha}} \sigma^\mu_{\beta\dot{\beta}} = 2\delta_{\beta\dot{\alpha}}\delta_{\dot{\beta}\alpha}. \tag{10.36}$$

The LHS can then be expressed as

$$g_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} = 2\sigma^0_{\alpha\dot{\alpha}}\sigma^0_{\beta\dot{\beta}} - \sum_\mu \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} = 2\delta_{\alpha\dot{\alpha}}\delta_{\beta\dot{\beta}} - 2\delta_{\alpha\dot{\beta}}\delta_{\beta\dot{\alpha}}, \tag{10.37}$$

where we used the fact that σ^0 is just the identity matrix and also the delta is symmetric upon indices exchange. This is exactly the Eq. (10.28). Thus, we have proven

$$g_{\mu\nu}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{10.38}$$

(b)

$$\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\mu\beta\dot{\beta}} = \frac{1}{2}g_{\delta\gamma}\sigma^\delta_{\alpha\dot{\alpha}}\sigma^\gamma_{\beta\dot{\beta}}\sigma^{\mu\beta\dot{\beta}} = \frac{1}{2}g_{\delta\gamma}\sigma^\delta_{\dot{\alpha}\alpha}\sigma^\gamma_{\beta\dot{\beta}}\sigma^{\mu\dot{\beta}\beta} = \frac{1}{2}g_{\delta\gamma}\sigma^\delta_{\dot{\alpha}\alpha}\,\mathrm{Tr}[\sigma^\gamma\sigma^\mu] = g_{\delta\gamma}\sigma^\delta_{\dot{\alpha}\alpha}\delta^{\gamma\mu} = \bar{\sigma}^\mu_{\dot{\alpha}\alpha}, \tag{10.39}$$

where we used the result of part (a) and Eq. (10.29).

10.4

(a) The Lorentz generator is defined by $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ from the Eq. (10.68) of the book. With the gamma matrices in the Majorana representation given by the Eq. (10.74) of the book, we can get

$$\begin{aligned}
J^3 = S^{12} &= \frac{i}{4}\big[\gamma^1,\gamma^2\big] = \frac{i}{4}\begin{pmatrix} 0 & -i\big[\sigma^3,\sigma^2\big] \\ i\big[\sigma^3,\sigma^2\big] & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \\
J^2 = -S^{13} &= -\frac{i}{4}\big[\gamma^1,\gamma^3\big] = -\frac{i}{4}\begin{pmatrix} \big[\sigma^3,\sigma^1\big] & 0 \\ 0 & \big[\sigma^3,\sigma^1\big] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \\
J^1 = S^{23} &= \frac{i}{4}\big[\gamma^2,\gamma^3\big] = \frac{i}{4}\begin{pmatrix} 0 & i\big[\sigma^2,\sigma^1\big] \\ i\big[\sigma^2,\sigma^1\big] & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \\
K^1 = S^{01} &= \frac{i}{4}\big[\gamma^0,\gamma^1\big] = \frac{i}{4}\begin{pmatrix} 0 & i\big[\sigma^2,\sigma^3\big] \\ i\big[\sigma^2,\sigma^3\big] & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \\
K^2 = S^{02} &= \frac{i}{4}\big[\gamma^0,\gamma^2\big] = \frac{i}{4}\begin{pmatrix} \big\{\sigma^2,\sigma^2\big\} & 0 \\ 0 & -\big\{\sigma^2,\sigma^2\big\} \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \\
K^3 = S^{03} &= \frac{i}{4}\big[\gamma^0,\gamma^3\big] = \frac{i}{4}\begin{pmatrix} 0 & -i\big[\sigma^2,\sigma^1\big] \\ -i\big[\sigma^2,\sigma^1\big] & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}.
\end{aligned} \tag{10.40}$$

(b)

- **Majorana Representation**
  From above, we have

$$\vec{J}^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 = \frac{1}{4}\begin{pmatrix} (\sigma^3)^2 + (\sigma^2)^2 + (\sigma^1)^2 & 0 \\ 0 & (\sigma^3)^2 + (\sigma^2)^2 + (\sigma^1)^2 \end{pmatrix} = \frac{3}{4}\mathbb{I}, \tag{10.41}$$

  where we used the fact that $\sigma^i\sigma^j = \frac{1}{2}\{\sigma^i,\sigma^j\} + \frac{1}{2}[\sigma^i,\sigma^j] = \delta^{ij} + \epsilon_{ijk}\sigma^k$ and so $(\sigma^i)^2 = \mathbf{1}$ (no Einstein summation here).

- **Left-handed Weyl Representation**
  From the Eq. (10.73) of the book, we have

$$\vec{J}^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 = \frac{1}{4}\begin{pmatrix} 1+1+1 & & & \\ & 1+1+1 & & \\ & & 1+1+1 & \\ & & & 1+1+1 \end{pmatrix} = \frac{3}{4}\mathbb{I}. \tag{10.42}$$

- **4-vector Representation** From the Eq. (10.14) of the book, we have

$$\vec{J}^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 = \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}. \tag{10.43}$$

For spin$-s$ particle, the eigenvalue of $\vec{J}^2$ is just $s(s+1)$. Therefore, the Majorana and the left-handed Weyl Representation describe a spin-half particle while the 4-vector representation corresponds to a spin-1 and a spin-0 degree of freedom.

(c)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix}\sigma^2\sigma^3\sigma^2\sigma^1 & 0 \\ 0 & -\sigma^2\sigma^3\sigma^2\sigma^1\end{pmatrix} = i\begin{pmatrix}-(\sigma^3)^T(\sigma^2)^2\sigma^1 & 0 \\ 0 & (\sigma^3)^T(\sigma^2)^2\sigma^1\end{pmatrix}$$
$$= i\begin{pmatrix}-\sigma^3\sigma^1 & 0 \\ 0 & \sigma^3\sigma^1\end{pmatrix} = \begin{pmatrix}\epsilon_{312}\sigma^2 & 0 \\ 0 & -\epsilon_{312}\sigma^2\end{pmatrix} = \begin{pmatrix}\sigma^2 & 0 \\ 0 & -\sigma^2\end{pmatrix}. \tag{10.44}$$

10.5

(a)

(b)

(c)

Chapter 11

Spinor solutions and CPT

11.1

(a)

$$\begin{aligned}(\gamma^5)^2 &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^2\gamma^3\gamma^2\gamma^3 \\ &= -\gamma^3\gamma^3 = \mathbb{I}\end{aligned} \tag{11.1}$$

(b)

$$\begin{aligned}\gamma_\mu \not{p} \gamma^\mu &= \gamma_\mu\gamma^\nu p_\nu \gamma^\mu = \gamma_\mu p_\nu(2g^{\nu\mu} - \gamma^\mu\gamma^\nu) \\ &= 2\not{p} - 2g_{\rho\mu}\gamma^\rho\gamma^\mu\not{p} \\ &= 2\not{p} - \frac{1}{2}(g_{\mu\alpha} + g_{\alpha\mu})\gamma^\alpha\gamma^\mu\not{p} \quad \text{(g is symmetric)} \\ &= 2\not{p} - \frac{1}{2}(g_{\mu\alpha}\gamma^\alpha\gamma^\mu + g_{\mu\alpha}\gamma^\mu\gamma^\alpha)\not{p} \quad \text{(Renaming the indices on the 2nd term in the parentheses)} \\ &= 2\not{p} - \frac{1}{2}g_{\mu\alpha}\{\gamma^\alpha, \gamma^\mu\}\not{p} \\ &= 2\not{p} - \frac{1}{2}g_{\mu\alpha}(2g^{\mu\alpha})\not{p} \\ &= 2\not{p} - 4\not{p} \\ &= -2\not{p}\end{aligned} \tag{11.2}$$

(c) We will first prove an identity that $\gamma_\mu\gamma^\nu\gamma^\alpha\gamma^\mu = 4g^{\nu\alpha}$.

$$\begin{aligned}\gamma_\mu\gamma^\nu\gamma^\alpha\gamma^\mu &= \gamma_\mu\gamma^\nu(2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha) \\ &= 2\gamma^\alpha\gamma^\nu + 2\gamma^\nu\gamma^\alpha \quad \text{(where we used the result from part b)} \\ &= 2\{\gamma^\alpha, \gamma^\nu\} \\ &= 4g^{\alpha\nu}\end{aligned} \tag{11.3}$$

$$\begin{aligned}
\gamma_\mu \not{p}\not{q}\not{p}\gamma^\mu &= p_\nu q_\alpha p_\beta(\gamma_\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu) \\
&= p_\nu q_\alpha p_\beta(\gamma_\mu\gamma^\nu\gamma^\alpha(2g^{\beta\mu} - \gamma^\mu\gamma^\beta)) \\
&= p_\nu q_\alpha p_\beta(2\gamma^\beta\gamma^\nu\gamma^\alpha - 4g^{\nu\alpha}\gamma^\beta), \quad \text{where we used eq.(11.3)} \\
&= p_\nu q_\alpha p_\beta(2\gamma^\beta(2g^{\nu\alpha} - \gamma^\alpha\gamma^\nu) - 4g^{\nu\alpha}\gamma^\beta) \\
&= p_\nu q_\alpha p_\beta(-2\gamma^\beta\gamma^\alpha\gamma^\nu) \\
&= -2\not{p}\not{q}\not{p}
\end{aligned} \tag{11.4}$$

(d)

$$\begin{aligned}
\{\gamma^5, \gamma^\mu\} &= i\{\gamma^0\gamma^1\gamma^2\gamma^3, \gamma^\mu\} \\
&= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3)
\end{aligned} \tag{11.5}$$

Each time the γ^μ anticommutes with a gamma matrix of different index, it introduces a (-1), and γ^μ commutes with itself. As $\mu = 0, 1, 2, 3$, if we want to move the γ^μ in the first term to the leftmost. It always introduces a $(-1)^3$. We thus arrived

$$\{\gamma^5, \gamma^\mu\} = i(-\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) = 0 \tag{11.6}$$

(e) We will first prove an identity that $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$.

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \frac{1}{2}(\mathrm{Tr}[\gamma^\mu\gamma^\nu] + \mathrm{Tr}[\gamma^\nu\gamma^\mu]) \quad \text{(trace is cyclic)} \\
&= \frac{1}{2}\,\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \\
&= \frac{1}{2}\,\mathrm{Tr}[\{\gamma^\mu, \gamma^\nu\}] \\
&= \frac{1}{2}\,\mathrm{Tr}[2g^{\mu\nu}] \\
&= 4g^{\mu\nu}
\end{aligned} \tag{11.7}$$

$$\begin{aligned}
\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big] &= \mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu(2g^{\beta\nu} - \gamma^\nu\gamma^\beta)\Big] \\
&= 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - \mathrm{Tr}\Big[\gamma^\alpha(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma^\beta\Big] \\
&= 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - 2g^{\mu\nu}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\beta\Big] + 2g^{\alpha\nu}\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\beta\Big] - \mathrm{Tr}\Big[\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\Big] \\
&= 2g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - 2g^{\mu\nu}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\beta\Big] + 2g^{\alpha\nu}\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\beta\Big] - \mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\Big] \\
&= g^{\beta\nu}\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu] - g^{\mu\nu}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\beta\Big] + g^{\alpha\nu}\,\mathrm{Tr}\Big[\gamma^\mu\gamma^\beta\Big] \\
&= 4g^{\beta\nu}g^{\alpha\mu} - 4g^{\mu\nu}g^{\alpha\beta} + 4g^{\alpha\nu}g^{\mu\beta} \quad \text{(eq.(11.7) is used)}
\end{aligned} \tag{11.8}$$

11.2

(a)

$$\begin{aligned}\sum_s u_s(p)\bar{u}_s(p) &= \sum_{s=1}^{2}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\xi_s\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\xi_s\end{pmatrix}^{\dagger}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ &= \begin{pmatrix}p\cdot\sigma & \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}\\ \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} & p\cdot\bar{\sigma}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & p\cdot\sigma\\ p\cdot\bar{\sigma} & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\end{pmatrix}\\ &= \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar{\sigma} & m\end{pmatrix}\\ &= \not{p}+m,\end{aligned}\tag{11.9}$$

where we used the fact that $\gamma^\mu = \begin{pmatrix} & \sigma^\mu\\ \bar{\sigma}^\mu & \end{pmatrix}$ in Weyl basis, $\xi_s\xi_s^\dagger = \mathbb{I}$, and $\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} = \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} = m$. Einstein summation rule is implied. Similarly,

$$\begin{aligned}\sum_s v_s(p)\bar{v}_s(p) &= \sum_{s=1}^{2}\begin{pmatrix}\sqrt{p\cdot\sigma}\eta_s\\ -\sqrt{p\cdot\bar{\sigma}}\eta_s\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\eta_s\\ -\sqrt{p\cdot\bar{\sigma}}\eta_s\end{pmatrix}^{\dagger}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ &= \begin{pmatrix}p\cdot\sigma & -\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}\\ -\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} & p\cdot\bar{\sigma}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\\ &= \begin{pmatrix}-\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & p\cdot\sigma\\ p\cdot\bar{\sigma} & -\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\end{pmatrix}\\ &= \begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar{\sigma} & -m\end{pmatrix}\\ &= \not{p}-m.\end{aligned}\tag{11.10}$$

(b)

$$\begin{aligned}\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) &= \begin{pmatrix}\sqrt{p\cdot\sigma}\xi_\sigma\\ \sqrt{p\cdot\bar{\sigma}}\xi_\sigma\end{pmatrix}^{\dagger}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma_\mu\\ \bar{\sigma}_\mu&0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_{\sigma'}\\ \sqrt{p\cdot\bar{\sigma}}\xi_{\sigma'}\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{p\cdot\sigma}\xi_\sigma\\ \sqrt{p\cdot\bar{\sigma}}\xi_\sigma\end{pmatrix}^{\dagger}\begin{pmatrix}\bar{\sigma}_\mu&0\\ 0&\sigma_\mu\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_{\sigma'}\\ \sqrt{p\cdot\bar{\sigma}}\xi_{\sigma'}\end{pmatrix}\\ &= \begin{pmatrix}\xi_\sigma\\ \xi_\sigma\end{pmatrix}^{\dagger}\begin{pmatrix}(p\cdot\sigma)\bar{\sigma}_\mu & 0\\ 0 & (p\cdot\bar{\sigma})\sigma_\mu\end{pmatrix}\begin{pmatrix}\xi_{\sigma'}\\ \xi_{\sigma'}\end{pmatrix}\\ &= 2\delta_{\sigma\sigma'}p^\mu,\end{aligned}\tag{11.11}$$

where we used the completeness relation of Pauli matrices $\mathrm{tr}((a\cdot\sigma)\bar{\sigma}) = 2a$ for any 4-vector a in the last line.

11.3

The key is to prove the interaction between the massless spin-1 particle and the spin-0 or spin-1/2 particle in the soft limit has the form $\frac{p\cdot\epsilon}{p\cdot q}$. Then, the derivation just follows what's in Section 9.5. Section 9.5 has already proves the general interaction form for spin-0, so we will just focus on the spin-1/2 case. By the Lorentz invariance, the only new interaction from the spin-1/2 case that has not yet already covered by the spin-0 case is the one that involves γ^μ. Suppose the interaction has the form $\gamma^\mu F(p,q)$, where F is the form factor that in general can depend on contractions of momenta or on contractions with γ-matrices.

Starting from Eq. (9.46) of the book, for an external leg of spin-1/2 particle of mass m, when tacked by a soft photon, the amplitude is modified to be

$$\begin{aligned}
\mathcal{M}_i(p_i,q) &= (-ieF_i)\epsilon^\mu \mathcal{M}_0(p_i-q)\frac{i(\not{p}-\not{q}+m)}{(p-q)^2-m^2}\gamma^\mu u(p_i) \\
&= -eF_i\mathcal{M}_0(p_i-q)\frac{\epsilon^\mu p^\nu\gamma^\nu\gamma^\mu - \not{q}\not{\epsilon} + m\not{\epsilon}}{2p\cdot q}u(p_i) \\
&= -eF_i\mathcal{M}_0(p_i-q)\frac{\epsilon^\mu p^\nu(2g^{\mu\nu}-\gamma^\mu\gamma^\nu) - \not{q}\not{\epsilon} + m\not{\epsilon}}{2p\cdot q}u(p_i) \\
&= -eF_i\mathcal{M}_0(p_i-q)\frac{2p\cdot\epsilon - \not{q}\not{\epsilon} - \not{\epsilon}(\not{p}-m)}{2p\cdot q}u(p_i) \\
&\approx -eF_i(0)\mathcal{M}_0(p_i)\frac{p\cdot\epsilon}{p\cdot q}u(p_i),
\end{aligned} \tag{11.12}$$

which has the intended interaction form. The derivation then follows exactly as that in Section 9.5. In the second line, we used the on shell conditions for the external spinor and the spin-1 particle. In the third line, we used the anticommutator of the gamma matrices. In the last line, we took the soft limit. Also the last term vanishes due to the equation of motion of the external spinor. The second term can be ignored in the sense that once one took the amplitude square, this term becomes $\mathrm{Tr}\left[\not{q}\not{\epsilon}\right] = 4q\cdot\epsilon = 0$ by the fact that the polarizations of a physical photon are transverse to their own momenta.

11.4

First notice that

$$\begin{aligned}
i\bar{u}(q)\frac{\sigma^{\mu\nu}(q_\nu-p_\nu)}{2m}u(p) &= -\bar{u}(q)\frac{[\gamma^\mu,\gamma^\nu](q_\nu-p_\nu)}{4m}u(p) \\
&= -\bar{u}(q)\frac{(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)(q_\nu-p_\nu)}{4m}u(p) \\
&= -\bar{u}(q)\frac{(g^{\mu\nu}-\gamma^\nu\gamma^\mu)q_\nu - (\gamma^\mu\gamma^\nu-g^{\mu\nu})p_\nu}{2m}u(p) \\
&= \bar{u}(q)\frac{\not{q}\gamma^\mu+\gamma^\mu\not{p}-q^\mu-p^\mu}{2m}u(p),
\end{aligned} \tag{11.13}$$

where we used the anticommutation of gamma matrices on the third line.

We then can have

$$\begin{aligned}\bar{u}(q)\left[\frac{q^\mu+p^\mu}{2m}+\frac{\not{q}\gamma^\mu+\gamma^\mu\not{p}-q^\mu-p^\mu}{2m}\right]u(p) &= \bar{u}(q)\frac{\not{q}\gamma^\mu+\gamma^\mu\not{p}}{2m}u(p)\\ &= \bar{u}(q)\frac{m\gamma^\mu+\gamma^\mu m}{2m}u(p)\\ &= \bar{u}(q)\gamma^\mu u(p),\end{aligned} \tag{11.14}$$

where we used the equation of motions for the spinor since they are on-shell, namely $\not{p}u(p) = mu(p)$ and $\bar{u}(q)\not{q} = \bar{u}(q)m$.

11.5

For a general Dirac matrix combination transformation Γ, we have a general spinor bilinear charge conjugation transformation as

$$\begin{aligned}C:\bar{\psi}\Gamma\psi \to (-i\gamma_2\psi)^T\gamma_0\Gamma(-i\gamma_2\psi^*) &= -\psi^T\gamma_2\gamma_0\Gamma\gamma_2\psi^*\\ &= -\psi_\alpha(\gamma_2\gamma_0\Gamma\gamma_2)_{\alpha\beta}\psi^*_\beta \quad \text{(where } \alpha,\ \beta \text{ are spinor indices)}\\ &= \psi^*_\alpha(\gamma_2\gamma_0\Gamma\gamma_2)_{\beta\alpha}\psi_\beta \quad \text{(anticommuting the spinors and relabeling } \alpha\to\beta\text{)}\\ &= \gamma^*(\gamma_2\gamma_0\Gamma\gamma_2)^T\psi\\ &= \gamma^*\gamma_0\gamma_0(\gamma_2\gamma_0\Gamma\gamma_2)^T\psi \quad (\gamma_0\gamma_0=\mathbb{I})\\ &= \bar{\psi}\gamma_0\gamma_2\Gamma^T\gamma_0\gamma_2\psi \quad (\gamma^T_{0,2}=\gamma_{0,2}).\end{aligned} \tag{11.15}$$

- For Eq. (11.54), $\Gamma=\gamma^5$:

$$\gamma_0\gamma_2(\gamma^5)^T\gamma_0\gamma_2 = (-1)^2\gamma_2\gamma^5\gamma_2 = (-1)^3(\gamma_2)^2\gamma^5 = (-1)^4\gamma^5 = \gamma^5. \tag{11.16}$$

  Thus, $C: i\bar{\psi}\gamma^5\psi \to i\bar{\psi}\gamma^5\psi$.

- For Eq. (11.55), $\Gamma=\gamma^5\gamma^\mu$:

$$\gamma_0\gamma_2(\gamma^5\gamma^\mu)^T\gamma_0\gamma_2 = \gamma_0\gamma_2(\gamma^\mu)^T\gamma^5\gamma_0\gamma_2. \tag{11.17}$$

  For $\mu=0,2$,

$$\gamma_0\gamma_2(\gamma^\mu)^T\gamma^5\gamma_0\gamma_2 = \gamma_0\gamma_2\gamma^\mu\gamma^5\gamma_0\gamma_2 = (-1)^2\gamma_0\gamma_2\gamma^\mu\gamma_0\gamma_2\gamma^5 = (-1)^3\gamma_0\gamma_2\gamma_0\gamma_2\gamma^\mu\gamma^5 = (-1)^5\gamma^\mu\gamma^5 = \gamma^5\gamma^\mu. \tag{11.18}$$

  For $\mu=1,3$,

$$\gamma_0\gamma_2(\gamma^\mu)^T\gamma^5\gamma_0\gamma_2 = -\gamma_0\gamma_2\gamma^\mu\gamma^5\gamma_0\gamma_2 = (-1)^5\gamma_0\gamma_2\gamma_0\gamma_2\gamma^\mu\gamma^5 = (-1)^7\gamma^\mu\gamma^5 = \gamma^5\gamma^\mu. \tag{11.19}$$

  Thus, $C: i\bar{\psi}\gamma^5\gamma^\mu\psi \to i\bar{\psi}\gamma^5\gamma^\mu\psi$.

- For Eq. (11.56), $\Gamma=\sigma^{\mu\nu}$:

$$\gamma_0\gamma_2(\sigma^{\mu\nu})^T\gamma_0\gamma_2 = -\gamma_0\gamma_2\left[\gamma^{\mu T},\gamma^{\nu T}\right]\gamma_0\gamma_2. \tag{11.20}$$

Notice for $\mu = \nu$, $\sigma^{\mu\nu} = 0$, and result holds trivially.

For $\mu = 0$ and $\nu = 2$ or $\mu = 2$ and $\nu = 0$,

$$-\gamma_0\gamma_2\left[\gamma^{\mu T}, \gamma^{\nu T}\right]\gamma_0\gamma_2 = -\gamma_0\gamma_2[\gamma^\mu, \gamma^\nu]\gamma_0\gamma_2 = (-1)^3\gamma_0\gamma_2\gamma_0\gamma_2[\gamma^\mu, \gamma^\nu] = (-1)^5[\gamma^\mu, \gamma^\nu] = -[\gamma^\mu, \gamma^\nu]. \tag{11.21}$$

For $\mu = 0, 2$ and $\nu = 1, 3$ or $\mu = 1, 3$ and $\nu = 0, 2$,

$$-\gamma_0\gamma_2\left[\gamma^{\mu T}, \gamma^{\nu T}\right]\gamma_0\gamma_2 = (-1)^2\gamma_0\gamma_2[\gamma^\mu, \gamma^\nu]\gamma_0\gamma_2 = (-1)^5\gamma_0\gamma_2\gamma_0\gamma_2[\gamma^\mu, \gamma^\nu] = (-1)^7[\gamma^\mu, \gamma^\nu] = -[\gamma^\mu, \gamma^\nu]. \tag{11.22}$$

Thus, $C : \bar{\psi}\sigma^{\mu\nu}\psi \to -\bar{\psi}\sigma^{\mu\nu}\psi$.

11.6

(a) First notice that as a projection operator, $P_{L/R}^2 = P_{L/R}$, $P_{L/R}P_{R/L} = 0$, and $P_L + P_R = 1$. Also, $P_{L/R}^\dagger = P_{L/R}$ since $\gamma^{5\dagger} = \gamma^5$, and $\gamma^\mu P_{L/R} = \gamma^\mu \frac{1\mp\gamma^5}{2} = \frac{1\pm\gamma^5}{2}\gamma^\mu = P_{R/L}\gamma^\mu$, where we have used $\left\{\gamma^5, \gamma^\mu\right\} = 0$. Then,

$$\begin{aligned}
\bar{\psi}\gamma^\mu\psi &= \psi^\dagger\gamma_0\gamma^\mu(P_L + P_R)\psi \\
&= \psi^\dagger\gamma_0\gamma^\mu(P_L^2 + P_R^2)\psi \\
&= \psi^\dagger\gamma_0 P_R\gamma^\mu P_L\psi + \psi^\dagger\gamma_0 P_L\gamma^\mu P_R\psi \\
&= \psi^\dagger P_L\gamma_0\gamma^\mu P_L\psi + \psi^\dagger P_R\gamma_0\gamma^\mu P_R\psi \\
&= (P_L\psi)^\dagger\gamma_0\gamma^\mu P_L\psi + (P_R\psi)^\dagger\gamma_0\gamma^\mu P_R\psi \\
&= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R.
\end{aligned} \tag{11.23}$$

This tells us that the QED vertex conserves chirality.

(b) From Eq. (11.25) of the book,

$$u_s = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, v_s = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_s \\ \sqrt{p\cdot\bar{\sigma}}\eta_s \end{pmatrix} = \sqrt{m}\begin{pmatrix} \eta_s \\ \eta_s \end{pmatrix}, \tag{11.24}$$

where we used the fact that in the rest limit, $\sqrt{p\cdot\sigma} = \sqrt{p\cdot\bar{\sigma}} = \sqrt{m}\mathbb{I}$.

With the Dirac matrices under Weyl representation from Eq. (10.64) of the book,

$$\gamma^0\gamma^i = \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix}, \tag{11.25}$$

it's easy to calculate the spinor current explicitly. For the time component,

$$\bar{\psi}_s\gamma^0\psi_{s'} = \psi_s^\dagger\psi_{s'}, \tag{11.26}$$

which clearly vanishes unless both spinors are spin up or down. This holds even if the spinor is not rest due to the normalization of the spinor. For the spatial components,

$$\begin{aligned}
\bar{u}_s(p)\gamma^i u_{s'}(p) &= m\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}^\dagger \begin{pmatrix} -\sigma_i & \\ & \sigma_i \end{pmatrix}\begin{pmatrix} \xi_{s'} \\ \xi_{s'} \end{pmatrix} \\
&= m(-\xi_s^\dagger\sigma_i\xi_{s'} + \xi_s^\dagger\sigma_i\xi_{s'}) = 0.
\end{aligned} \tag{11.27}$$

Similar calculation can show $\bar{v}_s(p)\gamma^i v_{s'}(p) = 0$ as well. The spatial component vanishes identically is clearly an expected result for non-relativistic limit. Notice currents like $\bar{u}_s\gamma^i v_{s'}$ or $\bar{v}_s\gamma^i u_{s'}$ are possible, but are relativistic effects, as these terms represent particle antiparticle annihilation and pair production.

(c) This is just to look at the Schrodinger-Pauli equation. From the Eq. (10.4) of the book, it's clear that only B field couples with the Pauli matrices that can change the spin.

(d) Since the spin is not changed while the spin has been flipped. The helicity has been flipped. On the other hand, chirality is not changed. This is consistent with (a), which only states the QED vertex conserves chirality, not helicity.

(e) One can use Stern-Gerlach experiment to measure the spin of a slow electron. In principle, one can simply send the electron through an inhomogeneous magnetic field and observing the deflection. However, since electron is charged particle, there is Lorentz force that will bend the trajectory in a circle. One can use electric field to balance this effect. Since just shown in (d) that the electric field can't alter the spin, this electric field will not affect the result.

(f) This is actually the famous Wu experiment, which establishes that parity is not conserved in weak interaction. One can measure the spin and momentum of the electron with respect to the colbalt-60 to find out the polarization. Notice since nickel-60 has almost the same rest mass as the colbalt-60. One can safely assume the nickel-60 is almost at rest and most of the momentum are carried out by the electron and the anti-neutrino, and thus the momentum of the electron and that of the anti-neutrino must be balanced off between themselves. As there is also not much energy available to the electron, it's safe to assume its speed is non-relativistic, and we can expect the spin is almost conserved in this case. If such decay is carried by a spin-1 gauge boson, due to the spin conservation of part (b)'s result, the electron and the anti-neutrino should either be both spin up or both spin down with respect to the colbalt-60's spin. On the other hand, momentum conservation says the two must have opposite momentum. These imply that the helicities of the two must be different. Also, as the neutrinos are almost massless, their chirality eigenstates almost correspond to their helicity eigenstates.

If the gauge boson responsible for the weak decay has an interaction vertex that conserves the chirality like the one in part (a), one would expect the electron emitted should be unpolarized with respect to the colbalt-60. In reality however, as the RH neutrino or LH anti-neutrino doesn't participate in any Standard Model interactions (and they are never observed at this point), the only anti-neutrinos can be emitted in this decay are all RH (both helicity and chirality sense, as neutrinos are almost massless), and the electrons have to be all in LH helicities to conserve spin. This establishes the fact that the weak interaction doesn't conserve parity. It's also interesting to know the parity is maximally broken in the weak interaction or in other words, it has a V-A interaction structure, so it doesn't interact with RH chirality particle and LH chirality anti-particle at all.

11.7

Throughout the following derivation, we maintain in the Weyl basis. From the Eq. (11.90) of the book, we can see

$$\text{C}\cdot\text{P}\cdot\text{T: } \psi^\dagger(x) \to (-\gamma_5\psi^*(-x))^\dagger = -\psi^T(-x)\gamma_5. \tag{11.28}$$

Under C · P · T, we have

- $\boldsymbol{\bar\psi\psi}$:

$$\begin{aligned}\bar\psi(x)\psi(x) &\to (-\psi^T(-x)\gamma_5)\gamma_0(-\gamma_5\psi^*(-x))\\ &= -\bar\psi(\gamma_5\gamma_0\gamma_5\gamma_0)^T\psi\\ &= \bar\psi(\gamma_5\gamma_5\gamma_0\gamma_0)^T\psi\\ &= \bar\psi(-x)\psi(-x).\end{aligned} \tag{11.29}$$

- $\boldsymbol{i\bar\psi\partial\!\!\!/\psi}$: Notice that $\partial_\mu \to -\partial_\mu$ and $i\partial_\mu \to (-i)(-\partial_\mu) = i\partial_\mu$. Thus, the operator $i\partial_\mu$ itself is invariant under CPT transformation

$$\begin{aligned}i\bar\psi(x)\partial\!\!\!/\psi(x) &\to (-1)^k i(-\psi^T(-x)\gamma_5)\gamma_0\gamma_\mu\partial_\mu(-\gamma_5\psi^*(-x))\\ &= (-1)^{k+1} i\partial_\mu\bar\psi(\gamma_5\gamma_0\gamma_\mu\gamma_5\gamma_0)^T\psi\\ &= (-1)^{k+2} i\bar\psi(\gamma_5\gamma_0\gamma_\mu\gamma_5\gamma_0)^T\partial_\mu\psi\\ &= (-1)^{k+2} i\bar\psi(\gamma_5\gamma_5\gamma_0\gamma_\mu\gamma_0)^T\partial_\mu\psi\\ &= (-1)^{k+2} i\bar\psi(\gamma_0\gamma_\mu\gamma_0)^T\partial_\mu\psi\\ &= (-1)^{k+2} i\bar\psi(\gamma_\mu^\dagger)^T\partial_\mu\psi\\ &= (-1)^{k+2} i\bar\psi\gamma_\mu^*\partial_\mu\psi\\ &= i\bar\psi(-x)\partial\!\!\!/\psi(-x),\end{aligned} \tag{11.30}$$

  where $k = 0$ for $\mu \neq 2$ and $k = 1$ for $\mu = 2$ since in the Weyl basis, γ_2 is imaginary. On the third line, we integrate by part and get an extra factor of -1.

- $\boldsymbol{\bar\psi A\!\!\!/\psi}$: This can be easily deduced from the above. Since $A_\mu(x) \to -A_\mu(-x)$, but there is no extra factor of -1 from the integration by part, and the rest parts just transform similarly as the above, we can deduce that $\bar\psi A\!\!\!/\psi$ is invariant under CPT transformation.

- $\boldsymbol{\bar\psi\gamma^\mu\gamma^5\psi W_\mu}$:

$$\begin{aligned}\bar\psi(x)\gamma^\mu\gamma^5\psi(x)W_\mu(x) &\to (-1)^k(-\psi^T(-x)\gamma^5)\gamma^0\gamma^\mu\gamma^5(-\gamma_5\psi^*(-x))(-W_\mu(-x))\\ &= (-1)^k\bar\psi(\gamma^5\gamma^0\gamma^\mu\gamma^5\gamma_5\gamma_0)^T\psi W_\mu\\ &= (-1)^k\bar\psi(\gamma^5\gamma^0\gamma^\mu\gamma_0)^T\psi W_\mu\\ &= (-1)^k\bar\psi(\gamma^5\gamma^{\mu\dagger})^T\psi W_\mu\\ &= (-1)^k\bar\psi\gamma^{\mu *}\gamma^5\psi W_\mu\\ &= \bar\psi(-x)\gamma^\mu\gamma^5\psi(-x)W_\mu(-x),\end{aligned} \tag{11.31}$$

  where again $k = 0$ for $\mu \neq 2$ and $k = 1$ for $\mu = 2$.

- $\boldsymbol{\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}}$: Notice that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to (-\partial_\mu)(-A_\nu) - (-\partial_\nu)(-A_\mu) = F_{\mu\nu}$. Thus the field tensor $F_{\mu\nu}$ itself is invariant under CPT transformation.

$$\begin{aligned}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x) &\to (-1)^k(-\psi^T(-x)\gamma^5)\gamma^0\sigma^{\mu\nu}(-\gamma_5\psi^*(-x))F_{\mu\nu}(-x)\\ &= (-1)^{k+1}\bar{\psi}(\gamma^5\gamma^0\sigma^{\mu\nu}\gamma_5\gamma_0)^T\psi F_{\mu\nu}\\ &= (-1)^{k+2}\bar{\psi}(\gamma^5\gamma_5\gamma^0\sigma^{\mu\nu}\gamma_0)^T\psi F_{\mu\nu}\\ &= (-1)^{k+2}\bar{\psi}(\gamma^0\sigma^{\mu\nu}\gamma_0)^T\psi F_{\mu\nu}\\ &= (-1)^{k+2}\bar{\psi}(\sigma^{\mu\nu\dagger})^T\psi F_{\mu\nu}\\ &= (-1)^{k+2}\bar{\psi}\sigma^{\mu\nu *}\psi F_{\mu\nu}\\ &= \bar{\psi}(-x)\sigma^{\mu\nu}\psi(-x)F_{\mu\nu}(-x),\end{aligned} \tag{11.32}$$

where now $k = 0$ for either $\mu = 2$ or $\nu = 2$ ($\sigma^{\mu\nu}$ is real in the Weyl basis) and $k = 1$ for $\mu \neq \nu \neq 2$ ($\sigma^{\mu\nu}$ is imaginary in the Weyl basis).

- $\boldsymbol{i\bar{\psi}\gamma^5\psi}$:

$$\begin{aligned}i\bar{\psi}(x)\gamma^5\psi(x) &\to (-i)(-\psi^T(-x)\gamma^5)\gamma^0\gamma^5(-\gamma_5\psi^*(-x))\\ &= i\bar{\psi}(\gamma^5\gamma^0\gamma^5\gamma_5\gamma_0)^T\psi\\ &= i\bar{\psi}(\gamma^5\gamma^0\gamma_0)^T\psi\\ &= i\bar{\psi}(-x)\gamma^5\psi(-x).\end{aligned} \tag{11.33}$$

Now, it's easy to see that terms like $(F_{\mu\nu})^n$ is invariant because $F_{\mu\nu}$ itself is invariant; $(\partial_\mu A_\mu)^n \to ((-\partial_\mu)(-A_\mu))^n = (\partial_\mu A_\mu)^n$ is invariant; $(A_\mu)^n(B_\mu)^n \to (-1)^{2n}(A_\mu)^n B_\mu^n \to A_\mu^n(B_\mu)^n$ is invariant (A_μ can be the same as the B_μ but the vector indices must contract properly to maintain the Lorentz invariance). The above derivations also illustrate that each vector index got a factor of -1 under CPT transformation then as long as these indices are properly contracted to maintain the Lorentz invariance, they must cancel out in pair and thus invariant under CPT transformation. Every derivative operator ∂_μ sandwiched between a spinor bilinear should be paired with a factor of i to cancel out the factor of -1 from integration by part.

It should be noted that any spinor bilinears can be decomposed into sum of any of the above terms. The reason is that in 4d spacetime, the Dirac spinors have 4 components, and thus the spinor bilinear can have at most 16 degrees of freedom. In the above derivations, the scalar bilinear $\bar{\psi}\psi$ takes 1 dof; the vector bilinear $\bar{\psi}\gamma^\mu\psi$ takes 4 dof; the anti-symmetric tensor bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$ takes 6 dof (since any general 4×4 anti-symmetric matrix can have at most 6 dof); the axial-vector bilinear $\bar{\psi}\gamma^\mu\gamma^5\psi$ takes 4 dof; the pseudo scalar bilinear $i\bar{\psi}\gamma^5\psi$ takes 1 dof (We shall justify in the Problem 11.8 that they are mutually orthogonal and thus, must take independent degree of freedom). In total, these spinor bilinears take $1+4+6+4+1 = 16$ dof. Therefore, the degree of freedom is exhausted and any bilinears that sandwiched more than five gamma matrices have at least two repeated gamma matrices and thus can be reduced and simplified into sums of any of the above terms.

Therefore, any Lorentz invariant terms one can write down in terms of Dirac spinors, γ-matrices, vector fields, and tensor fields are automatically invariant under CPT.

11.8

(a) First notice that by applying Eq. (10.142) onto Eq. (10.141) of the book, it's also true that $g_{\mu\nu}\bar{\sigma}^{\mu}_{\alpha_1\alpha_2}\bar{\sigma}^{\nu}_{\beta_1\beta_2} = 2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}$. Then, in the Weyl representation,

$$\bar{\psi}_1\gamma^\mu P_L\psi_2 = \begin{pmatrix}\psi^\dagger_{1L} & \psi^\dagger_{1R}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi_{2L}\\ \psi_{2R}\end{pmatrix} = \psi^\dagger_{1L}\bar{\sigma}^\mu\psi_{2L}. \tag{11.34}$$

Thus,

$$\begin{aligned}(\bar{\psi}_1\gamma^\mu P_L\psi_2)(\bar{\psi}_3\gamma_\mu P_L\psi_4) &= (\psi^\dagger_{1L}\bar{\sigma}^\mu\psi_{2L})(\psi^\dagger_{3L}\bar{\sigma}_\mu\psi_{4L})\\ &= (\psi^\dagger_{1L\alpha_1}(\bar{\sigma}^\mu)_{\alpha_1\alpha_2}\psi_{2L\alpha_2})(\psi^\dagger_{3L\beta_1}(\bar{\sigma}_\mu)_{\beta_1\beta_2}\psi_{4L\beta_2})\\ &= 2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}(\psi^\dagger_{1L\alpha_1}\psi_{2L\alpha_2})(\psi^\dagger_{3L\beta_1}\psi_{4L\beta_2})\\ &= -2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}(\psi^\dagger_{1L\alpha_1}\psi_{4L\beta_2})(\psi^\dagger_{3L\beta_1}\psi_{2L\alpha_2})\\ &= 2\varepsilon_{\alpha_1\beta_1}\varepsilon_{\beta_2\alpha_2}(\psi^\dagger_{1L\alpha_1}\psi_{4L\beta_2})(\psi^\dagger_{3L\beta_1}\psi_{2L\alpha_2})\\ &= (\psi^\dagger_{1L\alpha_1}(\bar{\sigma}^\mu)_{\alpha_1\beta_2}\psi_{4L\beta_2})(\psi^\dagger_{3L\beta_1}(\bar{\sigma}_\mu)_{\beta_1\alpha_2}\psi_{2L\alpha_2})\\ &= (\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma_\mu P_L\psi_2).\end{aligned} \tag{11.35}$$

The minus sign on the fourth line is due to the anti-commutativity of the spinors. In the fifth line, the minus sign is canceled because $\varepsilon_{\alpha_2\beta_2} = -\varepsilon_{\beta_2\alpha_2}$. **I believe the Schwartz has a typo that there should be no minus sign in front of the last equality because the spinors should anti-commute**. Also see Peskin & Schroeder's corrections comment on their p.51 [1].

(b) Since

$$\begin{aligned}\bar{\psi}_1\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_2 &= \begin{pmatrix}\psi^\dagger_{1L} & \psi^\dagger_{1R}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\alpha\\ \bar{\sigma}^\alpha & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\beta\\ \bar{\sigma}^\beta & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi_{2L}\\ \psi_{2R}\end{pmatrix}\\ &= \psi^\dagger_{1L}\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\psi_{2L}.\end{aligned} \tag{11.36}$$

Thus,

$$\begin{aligned}(\bar{\psi}_1\gamma^\mu\gamma^\alpha\gamma^\beta P_L\psi_2)(\bar{\psi}_3\gamma_\mu\gamma_\alpha\gamma_\beta P_L\psi_4) &= (\psi^\dagger_{1L}\bar{\sigma}^\mu\sigma^\alpha\bar{\sigma}^\beta\psi_{2L})(\psi^\dagger_{3L}\bar{\sigma}_\mu\sigma_\alpha\bar{\sigma}_\beta\psi_{4L})\\ &= (\psi^\dagger_{1L\alpha_1}\bar{\sigma}^\mu_{\alpha_1\alpha_2}\sigma^\alpha_{\alpha_2\alpha_3}\bar{\sigma}^\beta_{\alpha_3\alpha_4}\psi_{2L\alpha_4})(\psi^\dagger_{3L\beta_1}(\bar{\sigma}_\mu)_{\beta_1\beta_2}(\sigma_\alpha)_{\beta_2\beta_3}(\bar{\sigma}_\beta)_{\beta_3\beta_4}\psi_{4L\beta_4})\\ &= 8\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}\varepsilon_{\alpha_2\beta_2}\varepsilon_{\alpha_3\beta_3}\varepsilon_{\alpha_3\beta_3}\varepsilon_{\alpha_4\beta_4}(\psi^\dagger_{1L\alpha_1}\psi_{2L\alpha_4})(\psi^\dagger_{3L\beta_1}\psi_{4L\beta_4})\\ &= 32\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_4\beta_4}(\psi^\dagger_{1L\alpha_1}\psi_{2L\alpha_4})(\psi^\dagger_{3L\beta_1}\psi_{4L\beta_4})\\ &= -32\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_4\beta_4}(\psi^\dagger_{1L\alpha_1}\psi_{4L\beta_4})(\psi^\dagger_{3L\beta_1}\psi_{2L\alpha_4})\\ &= 32\varepsilon_{\alpha_1\beta_1}\varepsilon_{\beta_4\alpha_4}(\psi^\dagger_{1L\alpha_1}\psi_{4L\beta_4})(\psi^\dagger_{3L\beta_1}\psi_{2L\alpha_4})\\ &= 16(\psi^\dagger_{1L}\bar{\sigma}^\mu\psi_{4L})(\psi^\dagger_{3L}\bar{\sigma}_\mu\psi_{2L})\\ &= 16(\bar{\psi}_1\gamma^\mu P_L\psi_4)(\bar{\psi}_3\gamma_\mu P_L\psi_2).\end{aligned} \tag{11.37}$$

[1] *https://www.slac.stanford.edu/ mpeskin/QFT.html*

In the fourth line, we use the contraction relation of the Levi-Civita tensor: $\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = 2$. Again, the minus sign on the fifth line is due to the anti-commutativity of the spinors. In the sixth line, the minus sign is canceled because $\varepsilon_{\alpha_2\beta_2} = -\varepsilon_{\beta_2\alpha_2}$. **I believe the Schwartz has a typo that there should be no minus sign in front of the last equality because the spinors should anti-commute**.

(c) **There are again some typos in this question. It's should be $\mathbf{Tr}\Big[(\boldsymbol{\Gamma}^{\boldsymbol{M}})^\dagger\boldsymbol{\Gamma}^{\boldsymbol{N}}\Big] = 4\boldsymbol{\delta}^{\boldsymbol{MN}}$ instead of just $\mathbf{Tr}\Big[\boldsymbol{\Gamma}^{\boldsymbol{M}}\boldsymbol{\Gamma}^{\boldsymbol{N}}\Big] = 4\boldsymbol{\delta}^{\boldsymbol{MN}}$.** Otherwise, for example, $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$, which is 4 if $\mu = \nu = 0$ but -4 if $\mu = \nu = i$ and thus, cannot be properly normalized. We shall also use the set $\Gamma^M \in \{\mathbb{I}, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\gamma^\mu, i\gamma_5\}$ as the basis. The i in front of the pseudo-scalar basis is necessary for proper normalization as we will see (and also, when sandwiched between fermion bilinear, is required to keep CPT invariance as we showed in Problem 11.7).

We shall denote the **scalar** basis as **S**, the **vector** basis as **V**, the **tensor** basis as **T**, the **axial-vector** basis as **A**, and the **pseudo-scalar** basis as **P**.

Before moving on, we shall prove that $\gamma^0\Gamma^M\gamma^0 = (\Gamma^M)^\dagger$. For **S**, $\gamma^0\mathbb{I}\gamma^0 = (\gamma^0)^2 = \mathbb{I} = \mathbb{I}^\dagger$. For **V** and **T**, these are already proven in the Eq. (10.84) and Eq. (10.85) in the book. For **A**, $\gamma^0\gamma_5\gamma^\mu\gamma^0 = -\gamma_5\gamma^0\gamma^\mu\gamma^0 = -\gamma_5(\gamma^\mu)^\dagger = -\gamma_5^\dagger(\gamma^\mu)^\dagger = -(\gamma^\mu\gamma_5)^\dagger = (\gamma_5\gamma^\mu)^\dagger$. For **P**, $\gamma^0 i\gamma_5\gamma^0 = -i(\gamma^0)^2\gamma_5 = -i\gamma_5 = (i\gamma_5)^\dagger$

- **SS**:
$$\mathrm{Tr}\Big[\mathbb{I}^\dagger\mathbb{I}\Big] = 4. \tag{11.38}$$
- **SV**:
$$\mathrm{Tr}\Big[\mathbb{I}^\dagger\gamma^\mu\Big] = \mathrm{Tr}[\gamma^\mu] = 0, \tag{11.39}$$
by Eq. (A.39) of the book.
- **ST**:
$$\mathrm{Tr}\Big[\mathbb{I}^\dagger\sigma^{\mu\nu}\Big] = \mathrm{Tr}[\gamma^\mu\gamma^\nu] - \mathrm{Tr}[\gamma^\nu\gamma^\mu] = 0. \tag{11.40}$$
- **SA**:
$$\mathrm{Tr}\Big[\mathbb{I}^\dagger\gamma_5\gamma^\mu\Big] = \mathrm{Tr}[\gamma_5\gamma^\mu] = -\mathrm{Tr}[\gamma^\mu\gamma_5] = -\mathrm{Tr}[\gamma_5\gamma^\mu] = 0. \tag{11.41}$$
- **SP**:
$$\mathrm{Tr}\Big[\mathbb{I}^\dagger\gamma_5\Big] = \mathrm{Tr}[\gamma_5] = 0, \tag{11.42}$$
by Eq. (A.39) of the book.
- **VV**:
$$\begin{aligned}\mathrm{Tr}\Big[(\gamma^\mu)^\dagger\gamma^\nu\Big] &= \mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma^\nu\big] \\ &= \mathrm{Tr}\big[(2g^{0\mu} - \gamma^\mu\gamma^0)\gamma^0\gamma^\nu\big] \\ &= \mathrm{Tr}[2\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu] \\ &= 2g^{0\mu}\,\mathrm{Tr}\big[\gamma^0\gamma^\nu\big] - \mathrm{Tr}[\gamma^\mu\gamma^\nu] = 8g^{0\mu}g^{0\nu} - 4g^{\mu\nu} \\ &= \begin{cases} 4, & \text{if } \mu = \nu \\ 0, & \text{if } \mu \neq \nu \end{cases}.\end{aligned} \tag{11.43}$$

- **VT**:

$$\mathrm{Tr}\Big[(\gamma^\mu)^\dagger\sigma^{\alpha\beta}\Big] = \mathrm{Tr}\Big[\gamma^0\gamma^\mu\gamma^0\sigma^{\alpha\beta}\Big] = 0, \tag{11.44}$$

by Eq. (A.39) of the book since this has odd number of gamma matrices inside the trace.

- **VA**:

$$\begin{aligned}\mathrm{Tr}\Big[(\gamma^\mu)^\dagger\gamma_5\gamma^\nu\Big] &= -\,\mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma_5\gamma^\nu\gamma^\alpha\gamma^\alpha\big]\\ &= -\,\mathrm{Tr}\big[\gamma^\alpha\gamma^0\gamma^\mu\gamma^0\gamma_5\gamma^\nu\gamma^\alpha\big]\\ &= \mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma_5\gamma^\nu\gamma^\alpha\gamma^\alpha\big]\\ &= -\,\mathrm{Tr}\Big[(\gamma^\mu)^\dagger\gamma_5\gamma^\nu\Big]\\ &\implies \mathrm{Tr}\Big[(\gamma^\mu)^\dagger\gamma_5\gamma^\nu\Big] = 0,\end{aligned} \tag{11.45}$$

where we have chosen α such that $\alpha \neq 0$, $\alpha \neq \mu$, and $\alpha \neq \nu$. Since there are at most 3 different gamma matrices in the trace, it's always possible to choose a fourth gamma matrix γ^α that anti-commute with all others. Also notice that since $\alpha \neq 0$, $(\gamma^\alpha)^2 = -1$.

- **VP**:

$$\begin{aligned}\mathrm{Tr}\Big[(\gamma^\mu)^\dagger i\gamma_5\Big] &= i\,\mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma_5\big]\\ &= -i\,\mathrm{Tr}\big[\gamma_5\gamma^0\gamma^\mu\gamma^0\big]\\ &= -i\,\mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma_5\big]\\ &= -\,\mathrm{Tr}\Big[(\gamma^\mu)^\dagger i\gamma_5\Big]\\ &\implies \mathrm{Tr}\Big[(\gamma^\mu)^\dagger i\gamma_5\Big] = 0.\end{aligned} \tag{11.46}$$

- **TT**: By convention, for the tensor $\sigma^{\mu\nu}$ we can ask the indices $\mu < \nu$.

$$
\begin{aligned}
\mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger\sigma^{\alpha\beta}\Big] &= \mathrm{Tr}\Big[\gamma^0\sigma^{\mu\nu}\gamma^0\sigma^{\alpha\beta}\Big] \\
&= -\frac{1}{4}\,\mathrm{Tr}\Big[\gamma^0(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\gamma^0(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)\Big] \\
&= -\,\mathrm{Tr}\Big[\gamma^0\gamma^\mu\gamma^\nu\gamma^0\gamma^\alpha\gamma^\beta\Big] \\
&= -\,\mathrm{Tr}\Big[\gamma^0(2g^{\mu0}-\gamma^0\gamma^\mu)\gamma^\nu\gamma^\alpha\gamma^\beta\Big] \\
&= -2g^{\mu0}\,\mathrm{Tr}\Big[\gamma^0\gamma^\nu\gamma^\alpha\gamma^\beta\Big] + \mathrm{Tr}\Big[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\Big] \\
&= -8g^{\mu0}(-g^{0\alpha}g^{\nu\beta}+g^{0\beta}g^{\nu\alpha}) + 4(-g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}) \\
&= \begin{cases} 4g^{0\alpha}g^{\nu\beta}-4g^{0\beta}g^{\nu\alpha} & \mu=0 \\ -4g^{i\alpha}g^{\nu\beta}+4g^{i\beta}g^{\nu\alpha} & \mu=i\neq 0 \\ 0 & \mu\neq\alpha \end{cases} \\
&= \begin{cases} 4g^{0\alpha}g^{\nu\beta} & \mu=0 \\ -4g^{i\alpha}g^{\nu\beta} & \mu=i\neq 0 \\ 0 & \mu\neq\alpha \end{cases} \\
&= \begin{cases} 4g^{00}g^{ii} & \mu=0 \text{ (no sum on } i) \\ -4g^{ii}g^{jj} & \mu=i\neq 0, \nu=j\neq 0 \text{ (no sum on } i \text{ and } j) \\ 0 & \mu\neq\alpha \end{cases} \\
&= 4\delta^{\mu\alpha}\delta^{\nu\beta},
\end{aligned}
\tag{11.47}
$$

where we used the fact that it's always true that $\nu \neq 0$ and $\beta \neq 0$ and also, since $\mu < \nu$ and $\alpha < \beta$, it's impossible to have $\mu = \beta$ and $\nu = \alpha$.

- **TA**:

$$
\begin{aligned}
\mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger\gamma_5\gamma^\alpha\Big] &= \mathrm{Tr}\Big[\gamma_5(\sigma^{\mu\nu})^\dagger\gamma^\alpha\Big] \\
&= \mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger\gamma^\alpha\gamma_5\Big] \\
&= -\,\mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger\gamma_5\gamma^\alpha\Big] \\
&\implies \mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger\gamma_5\gamma^\alpha\Big] = 0.
\end{aligned}
\tag{11.48}
$$

- **TP**:

$$
\mathrm{Tr}\Big[(\sigma^{\mu\nu})^\dagger i\gamma_5\Big] = -\frac{1}{2}\,\mathrm{Tr}\big[\gamma^0(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\gamma^0\gamma_5\big] = 0 \tag{11.49}
$$

as already proven as a middle step in the case **VA**.

- **AA**:

$$
\begin{aligned}
\mathrm{Tr}\Big[(\gamma_5\gamma^\mu)^\dagger\gamma_5\gamma^\nu\Big] &= \mathrm{Tr}\big[\gamma^0\gamma_5\gamma^\mu\gamma^0\gamma_5\gamma^\nu\big] \\
&= \mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\gamma^\nu\big] \\
&= \mathrm{Tr}\big[\gamma^0(2g^{\mu 0} - \gamma^0\gamma^\mu)\gamma^\nu\big] \\
&= 2g^{\mu 0}\,\mathrm{Tr}\big[\gamma^0\gamma^\nu\big] - 4g^{\mu\nu} \\
&= 8g^{\mu 0}g^{0\nu} - 4g^{\mu\nu} \\
&= 4\delta^{\mu\nu}.
\end{aligned} \tag{11.50}
$$

- **AP**:

$$
\mathrm{Tr}\Big[(\gamma_5\gamma^\mu)^\dagger i\gamma_5\Big] = i\,\mathrm{Tr}\big[\gamma^0\gamma_5\gamma^\mu\gamma^0\gamma_5\big] = \mathrm{Tr}\big[\gamma^0\gamma^\mu\gamma^0\big] = 0, \tag{11.51}
$$

  by Eq. (A.39) of the book since this has odd number of gamma matrices inside the trace.

- **PP**:

$$
\mathrm{Tr}\Big[(i\gamma_5)^\dagger i\gamma_5\Big] = \mathrm{Tr}[\gamma_5\gamma_5] = \mathrm{Tr}[\mathbb{I}] = 4. \tag{11.52}
$$

We have thus proven that $\mathrm{Tr}\Big[(\Gamma^M)^\dagger\Gamma^N\Big] = 4\delta^{MN}$.

(d) We can write out the spinor indices of the bilinear product

$$
(\bar\psi_1\Gamma^M\psi_2)(\bar\psi_3\Gamma^N\psi_4) = (\bar\psi_1)_a\Gamma^M_{ab}(\psi_2)_b(\bar\psi_3)_c\Gamma^N_{cd}(\psi_4)_d. \tag{11.53}
$$

Thus, the spinor fields really just serve to bookkeeping the indices. As already argued in Problem 11.7 that the above set of 16 tensor structures can be used as a basis of 4×4 complex matrices. This means it's always possible to express

$$
(\bar\psi_1)_a\Gamma^M_{ab}(\psi_2)_b(\bar\psi_3)_c\Gamma^N_{cd}(\psi_4)_d = -\sum_{PQ} C^{MN}_{PQ}(\bar\psi_1)_a\Gamma^P_{ad}(\psi_4)_d(\bar\psi_3)_c\Gamma^Q_{cb}(\psi_2)_b, \tag{11.54}
$$

for some coefficients C^{MN}_{PQ} to be determined. **It's important to notice that there is a minus sign again coming from anti-commuting the spinor fields since their components are Grassmann number. Schwartz again missed the minus sign.** Since the spinor fields really just serve to bookkeeping the indices, this equation actually stands as

$$
\begin{aligned}
\Gamma^M_{ab}\Gamma^N_{cd} &= -\sum_{PQ} C^{MN}_{PQ}\Gamma^P_{ad}\Gamma^Q_{cb} \\
\Gamma^M_{ab}\Gamma^N_{cd}(\Gamma^{A\dagger})_{da}(\Gamma^{B\dagger})_{bc} &= -\sum_{PQ} C^{MN}_{PQ}\Gamma^P_{ad}\Gamma^Q_{cb}(\Gamma^{A\dagger})_{da}(\Gamma^{B\dagger})_{bc} \\
\mathrm{Tr}\Big[\Gamma^{A\dagger}\Gamma^M\Gamma^{B\dagger}\Gamma^N\Big] &= -\sum_{PQ} C^{MN}_{PQ}\,\mathrm{Tr}\Big[\Gamma^{A\dagger}\Gamma^P\Big]\,\mathrm{Tr}\Big[\Gamma^{B\dagger}\Gamma^Q\Big] \\
\mathrm{Tr}\Big[\Gamma^{A\dagger}\Gamma^M\Gamma^{B\dagger}\Gamma^N\Big] &= -16\sum_{PQ} C^{MN}_{PQ}\delta^{AP}\delta^{BQ} \\
C^{MN}_{PQ} &= -\frac{1}{16}\,\mathrm{Tr}\Big[\Gamma^{P\dagger}\Gamma^M\Gamma^{Q\dagger}\Gamma^N\Big],
\end{aligned} \tag{11.55}
$$

where in the second line, we multiply both sides by $(\Gamma^{A\dagger})_{da}(\Gamma^{B\dagger})_{bc}$.

Therefore, $(\bar{\psi}_1\Gamma^M\psi_2)(\bar{\psi}_3\Gamma^N\psi_4) = -\sum_{PQ}\frac{1}{16}\operatorname{Tr}\left[\Gamma^{P\dagger}\Gamma^M\Gamma^{Q\dagger}\Gamma^N\right]$.

11.9

(a)

(b)

(c)

(d)

(e)

(f)

Chapter 12

Spin and statistics

12.1

Assuming

$$\left[a_p^s, a_q^{s'\dagger}\right] = \left[b_p^s, b_q^{s'\dagger}\right] = (2\pi)^3\delta^3(p-q)\delta_{ss'} \tag{12.1}$$

Then,

$$\begin{aligned}\langle 0|\left[\bar{\psi}\psi(x), \bar{\psi}\psi(y)\right]|0\rangle &= \langle 0|[\bar{\psi}(x)\left[\psi(x), \bar{\psi}(y)\right]\psi(y) + \left[\bar{\psi}(x), \bar{\psi}(y)\right]\psi(x)\psi(y) \\ &+ \bar{\psi}(y)\bar{\psi}(x)[\psi(x), \psi(y)] + \bar{\psi}(y)\left[\bar{\psi}(x), \psi(y)\right]\psi(x)]|0\rangle \\ &= \langle 0|\left[\bar{\psi}(x)\left[\psi(x), \bar{\psi}(y)\right]\psi(y) - \bar{\psi}(y)\left[\psi(y), \bar{\psi}(x)\right]\psi(x)\right]|0\rangle \\ &= \langle 0|\bar{\psi}(x)[(i\partial\!\!\!/_x + m)D_1(t,r)]\psi(y)|0\rangle - \langle 0|\bar{\psi}(y)[(i\partial\!\!\!/_x + m)D_1(t,r)]\psi(x)|0\rangle\end{aligned} \tag{12.2}$$

where (12.87) of the book is used. We also used the fact that $D_1(t,r)$ is even under the PT transformation ($x \leftrightarrow y$) from (12.93) of the book. Then,

$$\begin{aligned}\langle 0|\left[\bar{\psi}\psi(x), \bar{\psi}\psi(y)\right]|0\rangle &= [(i\partial\!\!\!/_x + m)D_1(t,r)](\langle 0|\bar{\psi}(x)\psi(y)|0\rangle - \langle 0|\bar{\psi}(y)\psi(x)|0\rangle) \\ &= [(i\partial\!\!\!/_x + m)D_1(t,r)](\int \frac{d^3q}{(2\pi)^3}\frac{q\!\!\!/ - m}{2\omega_q}e^{-iq(x-y)} - \int \frac{d^3p}{(2\pi)^3}\frac{p\!\!\!/ - m}{2\omega_p}e^{ip(x-y)}),\end{aligned} \tag{12.3}$$

where we used (12.48) of the book. Notice that the momentum variable p and q are actually dummy, so we can relabel $p \to q$ in the second term above and arrives at

$$\begin{aligned}\langle 0|\left[\bar{\psi}\psi(x), \bar{\psi}\psi(y)\right]|0\rangle &= [(i\partial\!\!\!/_x + m)D_1(t,r)](\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}(q\!\!\!/ - m)(e^{-iq(x-y)} - e^{iq(x-y)}) \\ &= [(i\partial\!\!\!/_x + m)D_1(t,r)][(i\partial\!\!\!/_x - m)D(t,r)]\end{aligned} \tag{12.4}$$

As $D(t,r)$ vanish outside the lightcone, this must also vanish outside the lightcone. Since both $D(t,r)$ and $D_1(t,r)$ have support in the future and past lightcones, this will also have support there. Thus, the anticommutation relations $\left\{\bar{\psi}(x), \psi(y)\right\}$ is a sufficient but not necessary condition for the causality requirement $\left[\bar{\psi}\psi(x), \bar{\psi}\psi(y)\right] = 0$ outside the lightcone.

Chapter 13

Quantum electrodynamics

13.1

(a) The Moller scattering ($e^-e^- \to e^-e^-$) has two Feynman diagrams at tree level, the t channel and u channel. The channel is the same as the Rutherford scattering ($e^-p^+ \to e^-p^+$) with m_p being replaced by m_e, and a sign change on electric charge which will be squared out anyway. Since there is only one mass involved, we simply use m to denote it.

Modifying (13.83) of the book, we then have

$$\mathcal{M}_t = \frac{e^2}{t}\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2), \tag{13.1}$$

and from (13.91) of the book,

$$|\mathcal{M}_t|^2 = \frac{8e^4}{t^2}\left[u^2 + s^2 + 8tm^2 - 8m^4\right] \tag{13.2}$$

The u channel can be gotten from t channel by switching $p_3 \leftrightarrow p_4$. This change sends $s \to s$, $t \to u$, and $u \to t$. We then have

$$\mathcal{M}_u = \frac{e^2}{u}\bar{u}(p_4)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu u(p_2), \tag{13.3}$$

and

$$|\mathcal{M}_u|^2 = \frac{8e^4}{u^2}\left[t^2 + s^2 + 8um^2 - 8m^4\right] \tag{13.4}$$

To calculate the spin-averaged differential cross section, we will also need the cross terms of the matrix elements.

$$\mathcal{M}_t\mathcal{M}_u^\dagger = \frac{e^4}{tu}[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_\mu u(p_2)][\bar{u}(p_2)\gamma_\nu u(p_3)][\bar{u}(p_1)\gamma^\nu u(p_4)] \tag{13.5}$$

$$\mathcal{M}_t^\dagger\mathcal{M}_u = \frac{e^4}{tu}[\bar{u}(p_1)\gamma^\mu u(p_3)][\bar{u}(p_2)\gamma_\mu u(p_4)][\bar{u}(p_3)\gamma_\nu u(p_2)][\bar{u}(p_4)\gamma^\nu u(p_1)] \tag{13.6}$$

When averaging over the spins, the two cross terms actually give the same contribution. Thus,

$$\sum_{spin}(\mathcal{M}_t\mathcal{M}_u^\dagger + M_t^\dagger M_u) = \frac{2e^4}{tu}\,\mathrm{Tr}\Big[(\not{p}_3+m)\gamma^\mu(\not{p}_1+m)\gamma^\nu(\not{p}_4+m)\gamma_\mu(\not{p}_2+m)\gamma_\nu\Big] \tag{13.7}$$

Notice that only terms with even number of gamma matrices don't vanish from the trace, so the only possible surviving terms are either with 8 or 6 or 4 gamma matrices. Let's check each case.

With 8 γs, there is only one arrangement,

$$\begin{aligned}
\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\sigma\gamma_\nu\Big] &= -2\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\rho\gamma^\nu\gamma^\beta\gamma^\sigma\gamma_\nu\Big] && (\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\rho\gamma_\mu = -2\gamma^\rho\gamma^\nu\gamma^\beta)\\
&= -8g^{\beta\sigma}\,\mathrm{Tr}[\gamma^\alpha\gamma^\rho] && (\gamma^\nu\gamma^\beta\gamma^\sigma\gamma_\nu = 4g^{\beta\sigma})\\
&= -32g^{\beta\sigma}g^{\alpha\rho} && (\mathrm{Tr}[\gamma^\alpha\gamma^\rho] = 4g^{\alpha\rho})
\end{aligned} \tag{13.8}$$

With 6 γs,

$$\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma_\mu\gamma_\nu\Big] = 4\,\mathrm{Tr}\Big[\gamma^\alpha g^{\beta\nu}\gamma_\nu\Big] = 4\,\mathrm{Tr}\Big[\gamma^\alpha\gamma^\beta\Big] = 16g^{\alpha\beta} \tag{13.9}$$

Similarly, $\mathrm{Tr}\Big[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma_\mu\gamma_\nu\Big] = 16g^{\alpha\beta}$. All other arrangements of 6 γs are cyclic permutation and relabeling of the indices of these two and thus are all equal to $16g^{\alpha\beta}$.

With 4 γs, only one arrangement is possible,

$$\mathrm{Tr}\big[\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu\big] = -2\,\mathrm{Tr}\big[\gamma^\mu\gamma_\mu\big] = -8\,\mathrm{Tr}[I] = -32 \tag{13.10}$$

Collecting the terms,

$$\sum_{spin}(\mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_t^\dagger\mathcal{M}_u) = \frac{2e^4}{tu}[-32p_{12}p_{34} + 16m^2(p_{13}+p_{34}+p_{23}+p_{14}+p_{12}+p_{24}) - 32m^4], \tag{13.11}$$

where $p_{ij} \equiv p_i\cdot p_j$. Using the fact that all particles in Moller scattering have the same mass and from the (13.65)-(13.67) of the book, it's easy to see that $p_{12} = p_{34}$, $p_{13} = p_{24}$, and $p_{14} = p_{23}$. Thus,

$$\begin{aligned}
\sum_{spin}(\mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_t^\dagger\mathcal{M}_u) &= \frac{2e^4}{tu}[-32p_{12}^2 + 32m^2(p_{13}+p_{14}+p_{12}) - 32m^4]\\
&= -\frac{16e^4}{tu}[(s-2m^2)^2 - 2m^2(2m^2-t+2m^2-u+s-2m^2) + 4m^4]\\
&= -\frac{16e^4}{tu}[s^2 - 4m^2s + 8m^4 - 2m^2(2s-2m^2)] \qquad (s+t+u = \sum_i m_i^2)\\
&= -\frac{16e^4}{tu}(s^2 - 8m^2s + 12m^4)
\end{aligned} \tag{13.12}$$

Finally,

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{1}{4}\sum_{spin}|\mathcal{M}_t - \mathcal{M}_u|^2 = \frac{1}{4}\sum_{spin}(|\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 - \mathcal{M}_t\mathcal{M}_u^\dagger - \mathcal{M}_t^\dagger\mathcal{M}_u)$$
$$= \frac{2e^4}{t^2}(u^2+s^2+8m^2t-8m^4) + \frac{2e^4}{u^2}(t^2+s^2+8m^2u-8m^4) + \frac{4e^4}{tu}(s^2-8m^2s+12m^4) \tag{13.13}$$

With (5.33) of the book, in the CM frame,

$$(\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{256\pi^2E_{CM}^2}\sum_{spin}|\mathcal{M}|^2 \tag{13.14}$$

and substitute $E_{CM}^2 = s$. This is the spin-averaged differential cross section for Moller scattering.

(b) We take $p_1^\mu = (E,\vec{p}_i)$, $p_2^\mu = (E,-\vec{p}_i)$, $p_3^\mu = (E,\vec{p}_f)$, and $p_4^\mu = (E,-\vec{p}_f)$. We then have $\vec{p}_i\cdot\vec{p}_f = p^2\cos\theta$, where $p = |\vec{p}_i| = |\vec{p}_f| = \sqrt{\frac{E_{CM}}{2} - m^2}$ and θ is the scattering angle. Then,

$$\begin{aligned} s &= E_{CM}^2 = (p_1+p_2)^2 = 2m^2 + 2p_{12} = 2(m^2+E^2+p^2) = 4E^2 \\ t &= (p_1-p_3)^2 = 2m^2 - 2p_{13} = 2(m^2-E^2+p^2\cos\theta) = 2p^2(\cos\theta-1) \\ u &= (p_1-p_4)^2 = 2m^2 - 2p_{24} = 2(m^2-E^2-p^2\cos\theta) = -2p^2(\cos\theta+1) \end{aligned} \tag{13.15}$$

$$\begin{aligned} \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{2e^4}{4p^4(\cos\theta-1)^2}(4p^4(\cos\theta+1)^2 + 16E^4 + 16m^2p^2(\cos\theta-1) - 8m^4) \\ &+ \frac{2e^4}{4p^4(\cos\theta+1)^2}(4p^4(\cos\theta-1)^2 + 16E^4 - 16m^2p^2(\cos\theta+1) - 8m^4) \\ &+ \frac{4e^4}{4p^4\sin^2\theta}(16E^4 - 32m^2E^2 + 12m^4) \\ &= \frac{e^4}{p^4\sin^4\theta}[2p^4((\cos\theta+1)^4 + (\cos\theta-1)^4) + 8E^4((\cos\theta+1)^2 + (\cos\theta-1)^2 + 2\sin^2\theta) \\ &+ 8m^2p^2((\cos\theta-1)(\cos\theta+1)(\cos\theta+1-\cos\theta+1)) - 32m^2E^2\sin^2\theta \\ &- 4m^4((\cos\theta+1)^2 + (\cos\theta-1)^2 - 3\sin^2\theta)] \\ &= \frac{e^4}{p^4\sin^4\theta}[2p^4((\cos\theta+1)^4 + (\cos\theta-1)^4) + 32E^4 - 16m^2p^2\sin^2\theta \\ &- 32m^2E^2\sin^2\theta - 4m^4(5\cos^2\theta - 1)] \end{aligned} \tag{13.16}$$

In the NR limit, $p \ll E \approx m$, and we can pretty much ignore the term with factor of p^2

and replace E with m. Thus we can arrive at

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{p^4\sin^4\theta}[32E^4 - 32m^2E^2\sin^2\theta - 4m^4(5\cos^2\theta - 1)] \\ &= \frac{e^4m^4}{p^4\sin^4\theta}(32\cos^2\theta - 20\cos^2\theta + 4) \\ &= \frac{64\pi^2\alpha^2m^4}{p^4}\left(\frac{1+3\cos^2\theta}{\sin^4\theta}\right)\end{aligned} \tag{13.17}$$

$$(\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{256\pi^2E_{CM}^2}\sum_{spin}|\mathcal{M}|^2 = \frac{m^4\alpha^2}{E_{CM}^2p^4}\left(\frac{1+3\cos^2\theta}{\sin^4\theta}\right) \tag{13.18}$$

(c) I the UR limit, we can treat $m \approx 0$ and $E \approx p$. Then,

$$\begin{aligned}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4}{p^4\sin^4\theta}[2p^4((\cos\theta+1)^4 + (\cos\theta-1)^4) + 32E^4] \\ &= \frac{4e^4}{\sin^4\theta}(9 + 6\cos^2\theta + \cos^4\theta) \\ &= 64\pi^2\alpha^2\left(\frac{(3+\cos^2\theta)^2}{\sin^4\theta}\right)\end{aligned} \tag{13.19}$$

$$(\frac{d\sigma}{d\Omega})_{CM} = \frac{1}{256\pi^2E_{CM}^2}\sum_{spin}|\mathcal{M}|^2 = \frac{\alpha^2}{E_{CM}^2}\left(\frac{(3+\cos^2\theta)^2}{\sin^4\theta}\right) \tag{13.20}$$

13.2

The momenta of each particles have been written out in the Eq. (13.102) of the book. The electron is taken to be massless as this is very high energy limit.

$$p_4^\mu = p_1^\mu + p_2^\mu - p_3^\mu = (E - E' + m_p, \vec{p}_i - \vec{p}_f). \tag{13.21}$$

$$\begin{aligned}u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_p^2 - 2p_{14} = m_p^2 - 2p_{23}, \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 = -2p_{13} = 2m_p^2 - 2p_{24}, \\ s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 = m_p^2 + 2p_{12} = m_p^2 + 2p_{34}.\end{aligned} \tag{13.22}$$

and

$$\begin{aligned}p_{14} &= p_{23} = E'm_p, \\ p_{13} &= EE'(1-\cos\theta), \\ p_{24} &= m_p(E - E') + m_p^2 = m_p^2 + EE'(1-\cos\theta), \\ p_{12} &= p_{34} = Em_p,\end{aligned} \tag{13.23}$$

where θ is the scattering angle. Notice from the equation of p_{24}, we have

$$EE'(1-\cos\theta) = m_p(E - E'). \tag{13.24}$$

The scattering amplitude is

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{2e^4}{t^2}[u^2+s^2+4tm_p^2-2m_p^4] \\
&= \frac{2e^4}{4E^2E'^2(1-\cos\theta)^2}[(m_p^2-2p_{14})^2+(m_p^2+2p_{12})^2-8EE'(1-\cos\theta)m_p^2-2m_p^4] \\
&= \frac{2e^4}{4E^2E'^2(1-\cos\theta)^2}[-4E'm_p^3+4E'^2m_p^2+4Em_p^3+4E^2m_p^2-8EE'(1-\cos\theta)m_p^2] \\
&= \frac{e^4m_p^2}{2E^2E'^2\sin^4(\theta/2)}[-E'm_p+E'^2+Em_p+E^2-2EE'(1-\cos\theta)] \\
&= \frac{e^4m_p^2}{2E^2E'^2\sin^4(\theta/2)}[E'^2+E^2-EE'(1-\cos\theta)] \\
&= \frac{e^4m_p^2}{2E^2\sin^4(\theta/2)}[1+\frac{E^2}{E'^2}-\frac{E}{E'}(1-\cos\theta)],
\end{aligned} \tag{13.25}$$

where we have used Eq. (13.24) in the fourth line.

From problem 5.1,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2m_p}\left[E_4+E'(1-\frac{|\vec{p}_1|}{|\vec{p}_3|}\cos\theta)\right]^{-1}\frac{|\vec{p}_3|}{|\vec{p}_1|}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2m_p}\frac{E'}{E}\left[E-E'+m_p+E'-E\cos\theta\right]^{-1}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2m_p^2}\frac{E'}{E}\left[1+\frac{E}{m_p}(1-\cos\theta)\right]^{-1}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2m_p^2}\frac{E'}{E}\left[1+\frac{E-E'}{E'}\right]^{-1}|\mathcal{M}|^2 \\
&= \frac{e^4}{128\pi^2E^2\sin^4(\theta/2)}\frac{E'^2}{E^2}\left[1+\frac{E^2}{E'^2}-\frac{E}{E'}(1-\cos\theta)\right] \\
&= \frac{e^4}{128\pi^2E^2\sin^4(\theta/2)}\frac{E'}{E}\left[\frac{E'}{E}+\frac{E}{E'}-(1-\cos\theta)\right] \\
&= \frac{e^4}{128\pi^2E^2\sin^4(\theta/2)}\frac{E'}{E}\left[\frac{E'^2+E^2}{EE'}-2+2\cos^2(\theta/2)\right] \\
&= \frac{e^4}{128\pi^2E^2\sin^4(\theta/2)}\frac{E'}{E}\left[\frac{(E-E')^2}{EE'}+2\cos^2(\theta/2)\right] \\
&= \frac{e^4}{64\pi^2E^2\sin^4(\theta/2)}\frac{E'}{E}\left[\cos^2(\theta/2)+\frac{(E-E')}{m_p}\sin^2(\theta/2)\right], m_e\ll E,
\end{aligned} \tag{13.26}$$

where Eq. (13.24) has been used again in the third to fourth line as well as the second to the last line to the last line.

13.3

(a) Actually, the formula only applies for the rest frame of the mother particle. In the rest frame of the mother particle,

$$p_1 = (m_\phi, 0), p_2 = (E, \vec{p}_2), p_3 = (E, -\vec{p}_2). \tag{13.27}$$

Since $p_1 = p_2 + p_3$, squaring both sides tells us $m_\phi^2 = 2m_e^2 + 2(E^2 + p_f^2) = 4m_e^2 + 4p_f^2$ or

$$p_f = \frac{\sqrt{m_\phi^2 - 4m_e^2}}{2} = \frac{m_\phi\sqrt{1-4x^2}}{2}, \tag{13.28}$$

where $p_f = |\vec{p}_2|$ The phase-space integral becomes

$$d\Gamma = \frac{1}{8m_\phi}|\mathcal{M}|^2 \frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}\frac{1}{E^2}(2\pi)^4\delta^4(p_1 - p_2 - p_3). \tag{13.29}$$

Integrate over p_3,

$$\begin{aligned} \Gamma &= \frac{1}{32\pi^2 m_\phi}|\mathcal{M}|^2 \int dp_f d(\cos\theta)d\phi \frac{p_f^2}{E^2}\delta(2E - m_\phi) \\ &= \frac{1}{8\pi m_\phi}|\mathcal{M}|^2 \int dp_f \frac{p_f^2}{E^2}\delta(2\sqrt{m_e^2 + p_f^2} - m_\phi) \\ &= \frac{1}{16\pi m_\phi}|\mathcal{M}|^2 \int_{m_\phi - 2m_e}^{\infty} dx \frac{\frac{m_\phi\sqrt{1-4x^2}}{2}}{\frac{m_\phi}{2}}\delta(x) \\ &= \frac{\sqrt{1-4x^2}}{16\pi m_\phi}|\mathcal{M}|^2\theta(m_\phi - 2m_e). \end{aligned} \tag{13.30}$$

The θ function of course just tells us the mass of the particle ϕ needs to be at least two times as much as the electron mass for the decay to occur.

(b)
- Scalar

  In the case of a scalar, the decay amplitude is given by

$$i\mathcal{M}_S = ig_S\bar{u}(p_2)v(p_3), \tag{13.31}$$

  so

$$|\mathcal{M}_S|^2 = g_S^2[\bar{u}(p_2)v(p_3)][\bar{v}(p_3)u(p_2)]. \tag{13.32}$$

  Dividing the amplitude by a factor of two to account for taking unpolarized measurements for the decay products, and taking the spin sum,

$$\begin{aligned} \frac{1}{2}\sum_{spins}|\mathcal{M}_S|^2 &= \frac{1}{2}g_S^2\,\mathrm{Tr}\left[(\not{p}_2 + m_e)(\not{p}_3 - m_e)\right] \\ &= g_S^2(2p_{23} - 2m_e^2) \\ &= g_S^2 m_\phi^2(1-4x^2), \end{aligned} \tag{13.33}$$

where we used $p_{23} = (p_2 + p_3)^2/2 - m_e^2 = m_\phi^2/2 - m_e^2$. Notice unlike the scattering amplitude, the decay amplitude square is not dimensionless, as it should be to cancel out the inverse mass dimension from the phase space integral and give the whole decay rate a mass dimension 1. We have

$$\Gamma_S = \frac{g_S^2 m_\phi (1-4x^2)^{\frac{3}{2}}}{16\pi}. \tag{13.34}$$

This has the correct mass dimension $[\Gamma_S] = 1$.

- Pseudoscalar

  In the case of a pseudoscalar, the decay amplitude is given by

$$i\mathcal{M}_P = -g_P \bar{u}(p_2)\gamma_5 v(p_3), \tag{13.35}$$

  Also,

$$\mathcal{M}_P^\dagger = -ig_P[\bar{u}(p_2)\gamma_5 v(p_3)]^\dagger = -ig_P v(p_3)^\dagger \gamma_5 \gamma^0 u(p_2) = ig_P v(p_3)^\dagger \gamma^0 \gamma_5 u(p_2) = ig_P \bar{v}(p_3)\gamma_5 u(p_2). \tag{13.36}$$

  so

$$|\mathcal{M}_P|^2 = -g_P^2[\bar{u}(p_2)\gamma_5 v(p_3)][\bar{v}(p_3)\gamma_5 u(p_2)] \tag{13.37}$$

  Dividing the amplitude by a factor of two to account for taking unpolarized measurements for the decay products, and taking the spin sum,

$$\begin{aligned}\frac{1}{2}\sum_{spins}|\mathcal{M}_P|^2 &= -\frac{1}{2}g_P^2 \operatorname{Tr}\left[(\not{p}_2 + m_e)\gamma_5(\not{p}_3 - m_e)\gamma_5\right] \\ &= -g_P^2(-2p_{23} - 2m_e^2) \\ &= g_P^2 m_\phi^2.\end{aligned} \tag{13.38}$$

  Thus,

$$\Gamma_P = \frac{g_P^2 m_\phi \sqrt{1-4x^2}}{16\pi}. \tag{13.39}$$

- Vector

  In the case of a vector, the decay amplitude is given by

$$i\mathcal{M}_V = ig_V \bar{u}(p_2)\gamma^\mu v(p_3)\epsilon_1^\mu, \tag{13.40}$$

  so

$$|\mathcal{M}_V|^2 = g_V^2[\bar{u}(p_2)\gamma^\mu v(p_3)][\bar{v}(p_3)\gamma^\nu u(p_2)]\epsilon_1^\mu \epsilon_1^{\nu *}. \tag{13.41}$$

  Dividing the amplitude by a factor of two to account for taking unpolarized measurements for the decay products, and another factor of three from averaging out the polarization states of the incoming vector boson, and taking the spin sum,

$$\begin{aligned}\frac{1}{6}\sum_{spins,pols}|\mathcal{M}_V|^2 &= -\frac{1}{6}g_V^2 \operatorname{Tr}\left[(\not{p}_2 + m_e)\gamma^\mu(\not{p}_3 - m_e)\gamma^\nu\right]g^{\mu\nu} \\ &= \frac{1}{6}g_V^2(8p_{23} + 16m_e^2) \\ &= \frac{2}{3}g_V^2 m_\phi^2(1+2x^2)\end{aligned} \tag{13.42}$$

where we have replaced the polarization sum of the vector boson in the first line with $\sum_{pols.i}\epsilon^i_\mu\epsilon^i_\nu \to -g_{\mu\nu}$ from the Eq. (13.112) of the book. Thus,

$$\Gamma_V = \frac{g_V^2 m_\phi(1+2x^2)\sqrt{1-4x^2}}{24\pi}. \tag{13.43}$$

- Axial vector

  In the case of an axial vector, the decay amplitude is given by

$$i\mathcal{M}_A = -g_A\bar{u}(p_2)\gamma^\mu\gamma_5 v(p_3)\epsilon_1^\mu, \tag{13.44}$$

Also,

$$\mathcal{M}_A^\dagger = -ig_A v(p_3)^\dagger\gamma_5\gamma^{\nu\dagger}\gamma^0 u(p_2)\epsilon_1^{\nu*} = -ig_A v(p_3)^\dagger\gamma_5\gamma^0\gamma^\nu u(p_2)\epsilon_1^{\nu*} = ig_A\bar{v}(p_3)\gamma_5\gamma^\nu u(p_2)\epsilon_1^{\nu*}, \tag{13.45}$$

where we used $\gamma^{\nu\dagger} = \gamma^0\gamma^\nu\gamma^0$, so

$$|\mathcal{M}_A|^2 = -g_A^2[\bar{u}(p_2)\gamma^\mu\gamma_5 v(p_3)][\bar{v}(p_3)\gamma_5\gamma^\nu u(p_2)]\epsilon_1^\mu\epsilon_1^{\nu*}. \tag{13.46}$$

Dividing the amplitude by a factor of two to account for taking unpolarized measurements for the decay products, and another factor of three from averaging out the polarization states of the incoming vector boson, and taking the spin sum,

$$\begin{aligned}\frac{1}{6}\sum_{spins,pols}|\mathcal{M}_A|^2 &= \frac{1}{6}g_A^2\,\mathrm{Tr}\left[(\not{p}_2+m_e)\gamma^\mu\gamma_5(\not{p}_3-m_e)\gamma_5\gamma^\nu\right]g^{\mu\nu}\\ &= \frac{1}{6}g_A^2(8p_{23}-16m_e^2)\\ &= \frac{2}{3}g_A^2 m_\phi^2(1-6x^2).\end{aligned} \tag{13.47}$$

Thus,

$$\Gamma_A = \frac{g_A^2 m_\phi(1-6x^2)\sqrt{1-4x^2}}{24\pi}. \tag{13.48}$$

(c) First we should realize that $m_e, m_\mu \ll m_\phi = 4$ GeV, so $x_e \approx x_\mu \approx 0$. As such, $\Gamma(\phi \to e^+ + e^-) \approx \Gamma(\phi \to \mu^+ + \mu^-) = (1 - \Gamma(\phi \to \tau^+ + \tau^-))/2 = 0.375$. We can use the ratio of two decay rates to cancel out the unknown coupling constant.

If ϕ is a scalar,

$$\frac{\Gamma_\tau}{\Gamma_e} = (1-4x_\tau^2)^{\frac{3}{2}} = (1-4(1.776/4)^2)^{\frac{3}{2}} = 0.0972. \tag{13.49}$$

If ϕ is a pseudoscalar,

$$\frac{\Gamma_\tau}{\Gamma_e} = \sqrt{1-4x_\tau^2} = \sqrt{1-4(1.776/4)^2} = 0.460. \tag{13.50}$$

If ϕ is a vector,

$$\frac{\Gamma_\tau}{\Gamma_e} = (1+2x_\tau^2)\sqrt{1-4x_\tau^2} = (1+2(1.776/4)^2)\sqrt{1-4(1.776/4)^2} = 0.641. \tag{13.51}$$

The ϕ can not be an axial vector, as it needs the daughter particles to satisfy $6x^2 < 1$ for the decay to happen, as a negative decay rate is not physical.

Since $\frac{\Gamma_\tau}{\Gamma_e} = 0.25/0.375 = 0.667$, this is closest to a prediction of vector boson, and thus, the spin and parity of ϕ is $J^P = 1^-$, which means spin 1 and odd parity.

13.4

The book has already proved the case with 2 photon fields attached with the fermion loop. One can also think about a tadpole diagram although this diagram vanishes for QED, as the photon field has zero vacuum expectation value, but the statement that there is -1 for each fermion loop is still true even if the external fields attached to the fermion loop is not gauge boson fields but scalar fields. So to be more rigorous, one should consider a tadpole diagram, and observe its time-ordered production

$$G_1 = (ig) \langle 0| T\{\phi^{\alpha_1\beta_1\cdots}(x_1)\bar{\psi}_a(x)\Gamma^{\alpha\beta\cdots}_{ab}\phi^{\alpha\beta\cdots}(x)\psi_b(x)\} |0\rangle \,, \tag{13.52}$$

where g is a general coupling constant, ϕ is a general tensor field, and Γ a general tensor structure to encode the interaction between the spinors and the tensor field. We used Greek letter to denote the tensor indices while Latin letter for the spinor indices. It's always true that there is exactly one boson field sandwiched between the spinor bilinear, as this is the only way that the interaction has a mass dimension 4. One needs to move the $\psi_b(x)$ to the left of the $\bar{\psi}_a(x)$ to achieve the correct order that the field that is created is immediately destroyed. That is simply

$$...\bar{\psi}_a(x)...\psi_b(x) = -...\psi_b(x)\bar{\psi}_a(x)..., \tag{13.53}$$

as everything else in the time-ordered product just commuted with the spinor fields and thus are not relevant.

Now for n external tensor fields attached with the fermion loop, again, as spinor fields simply commute with everything inside the time-ordered product except with other spinor fields, thus, everything other than the spinor fields are not relevant for the discussion,

$$G_n = (ig)^n \langle 0| T\{...\bar{\psi}_{a_1}(x_1)...\psi_{b_1}(x_1)\bar{\psi}_{a_2}(x_2)...\psi_{b_{n-1}}(x_{n-1})\bar{\psi}_{a_n}(x_n)...\psi_{b_n}(x_n)\}. \tag{13.54}$$

Every spinor is already in the right order except for the first one and the last one, where the fermion loop connects back to the origin and the $\psi_{b_n}(x_n)$ needs to annihilate the $\psi_{a_1}(x_1)$.

$$G_n = (ig)^n(-1)^{2n-1} \langle 0| T\{...\psi_{b_n}(x_n)\bar{\psi}_{a_1}(x_1)...\psi_{b_1}(x_1)\bar{\psi}_{a_2}(x_2)...\psi_{b_{n-1}}(x_{n-1})\bar{\psi}_{a_n}(x_n)...\}. \tag{13.55}$$

As $(-1)^{2n-1} = -1$, we always get a -1 from each fermion loop.

13.5

(a) The order of the diagram is at $\mathcal{O}(e^4)$. There are 2 box diagrams at this order, as shown in Fig. 13.1. Then, there are 8 diagrams that has a tadpole loop attached onto an external spinor leg (4 with an electron loop and 4 with a muon loop), shown in Fig. 13.2 as one example. Then, there are 4 diagrams that each corrects one of the external propagators,

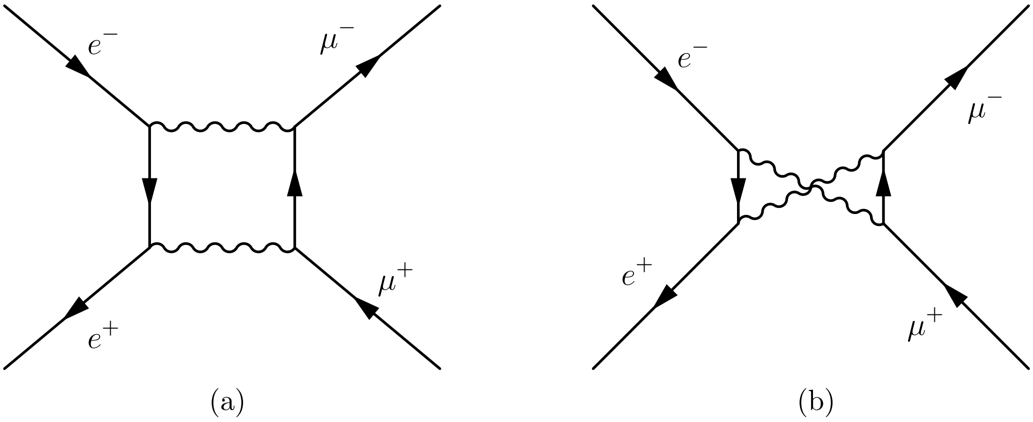


Fig. 13.1: $\mathcal{O}(e^4)$ box diagrams for $e^+e^- \to \mu^+\mu^-$.

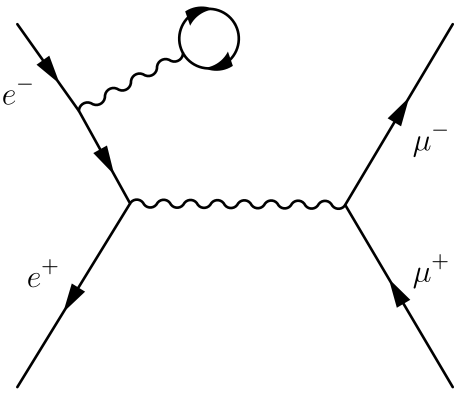


Fig. 13.2: An example of $\mathcal{O}(e^4)$ diagrams for $e^+e^- \to \mu^+\mu^-$ with a tadpole attached onto the external spinor leg.

shown in Fig. 13.3 as one example. There are 2 diagrams that has vacuum polarization correction for the intermediate photon propagator (1 with an electron loop and 1 with a muon loop), as shown in Fig. 13.4. Lastly, there are 2 diagrams that involve a vertex correction, as shown in Fig. 13.5. Therefore, there are in total $2 + 8 + 4 + 2 + 2 = 18$ diagrams at $\mathcal{O}(e^4)$ order.

(b) Gauge invariance means the diagram should be independent of the gauge choice ξ variable in the photon propagator. There are two internal photon propagators in the graph. One can effectively treat this diagram as the t-channel diagrams as Eq. (9.41) of the book (replacing the scalars with the spinors). Thus, the diagram is gauge invariant when also including the diagrams with the photon lines like a u-channel diagram, which is exactly the diagram shown in the Fig. 13.1b (no 4-vertex diagram in spinor QED).

(c) Let p_1 (p_2) to denote the incoming electron (positron)'s four-momentum. Let q (k) to denote the top (bottom) photon line's four-momentum. We can ignore the final states and parameterize that the photon lines attached to a generic tensor $X_{\alpha\beta}$ just like the procedure

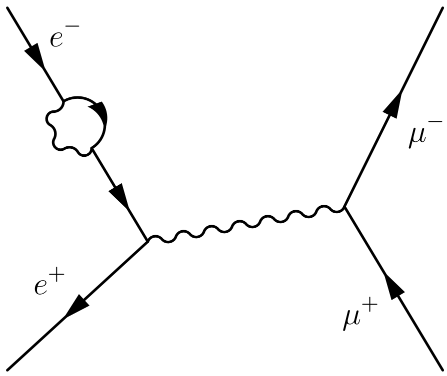


Fig. 13.3: An example of $\mathcal{O}(e^4)$ diagrams for $e^+e^- \to \mu^+\mu^-$ that has external propagator correction for one of the legs.

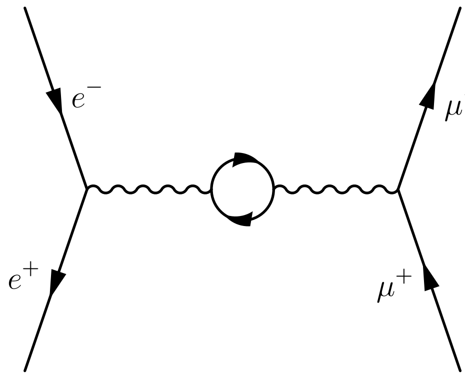


Fig. 13.4: $\mathcal{O}(e^4)$ diagrams for $e^+e^- \to \mu^+\mu^-$ of which the intermediate photon propagator has vacuum polarization correction.

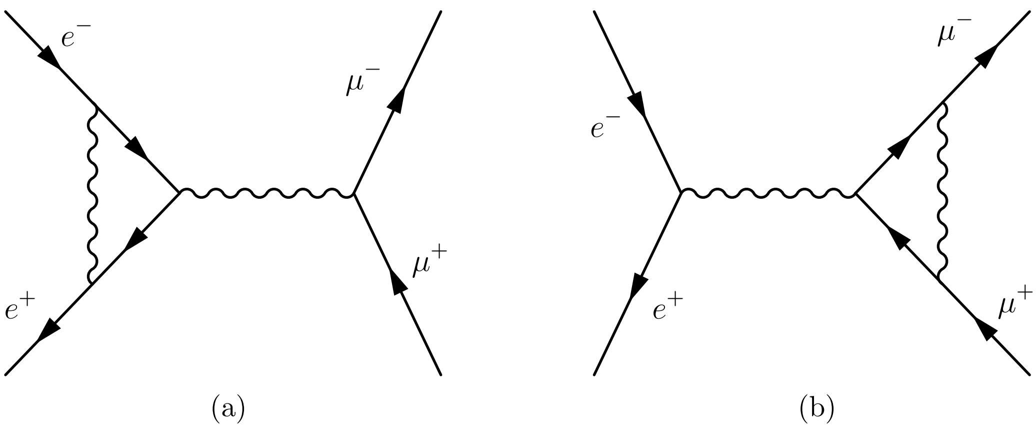


Fig. 13.5: $\mathcal{O}(e^4)$ vertex correction diagrams for $e^+e^- \to \mu^+\mu^-$.

in the Section 9.4 of the book. Then,

$$\begin{aligned}\mathcal{M}_t = \int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)(-ie)^2 \\ \times \bar{v}(p_2)\gamma^\nu\frac{(\not{p}_1-\not{q}+m_e)}{(p_1-q)^2-m_e^2}\gamma^\mu u(p_1)\Pi_{\mu\alpha}(q)\Pi_{\nu\beta}X_{\alpha\beta}(q,k),\end{aligned} \tag{13.56}$$

where $\Pi_{\mu\alpha}(q) = -\frac{i}{q^2}\left[g_{\mu\alpha}-(1-\xi)\frac{q_\mu q_\alpha}{q^2}\right]$. We can use the fact that the initial spinors are on-shell such that they satisfy the Dirac equations:

$$\not{p}_1 u(p_1) = m_e u(p_1), \tag{13.57}$$

$$\bar{v}(p_2)\not{p}_2 = m_e\bar{v}(p_2). \tag{13.58}$$

Replacing $\Pi_{\mu\alpha}\to\xi q_\mu q_\alpha$, we find

$$\begin{aligned}\mathcal{M}_t &= -\xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\ &\qquad\times \bar{v}(p_2)\gamma^\nu\frac{(\not{p}_1-\not{q}+m_e)\not{q}}{(p_1-q)^2-m_e^2}u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k) \\ &= -\xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\ &\qquad\times \bar{v}(p_2)\gamma^\nu\frac{(2p_1\cdot q-\not{q}\not{p}_1-q^2+m_e\not{q})}{-2p_1\cdot q+q^2}u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k) \\ &= \xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)\bar{v}(p_2)\gamma^\nu u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k),\end{aligned} \tag{13.59}$$

where in the second equality, we used the fact $\not{p}_1\not{q} = p_1^\mu q^\nu\gamma^\mu\gamma^\nu = p_1^\mu q^\nu(2g^{\mu\nu}-\gamma^\nu\gamma^\mu) = 2p_1\cdot q-\not{q}\not{p}_1$, $\not{q}\not{q}=q^2$, and $p_1^2=m_e^2$. In the third equality, we used the Dirac equation. Similarly, we can have

$$\begin{aligned}\mathcal{M}_u &= -\xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\ &\qquad\times \bar{v}(p_2)\frac{\not{q}(\not{q}-\not{p}_2+m_e)}{(q-p_2)^2-m_e^2}\gamma^\nu u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k) \\ &= -\xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q) \\ &\qquad\times \bar{v}(p_2)\frac{(q^2-2p_2\cdot q+\not{p}_2\not{q}+m_e\not{q})}{-2p_2\cdot q+q^2}\gamma^\nu u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k) \\ &= -\xi e^2\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-k-q)\bar{v}(p_2)\gamma^\nu u(p_1)q^\alpha\Pi_{\nu\beta}X_{\alpha\beta}(q,k).\end{aligned} \tag{13.60}$$

Now, this is just opposite to the $\mathcal{M}_t$ channel. Thus, $\mathcal{M}_t+\mathcal{M}_u=0$, which is exactly what the gauge invariance required. Notice we only required the initial fermions to be on-shell without ever invoking whether the photons are on-shell or not.

13.6

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Chapter 14

Path integrals

14.1

We will use $\phi(x) =: \phi_x$, $J(x) =: J_x$, $M(x,y) =: M_{xy}$, and $\delta^4(x-y) =: \delta_{xy}$ to save notation.

$$\begin{aligned}
&\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y[\phi_x^* M_{xy}\phi_y] + i\int d^4x[J_x^*\phi_x + \phi_x^* J_x]\right\} \\
&= \int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y\left[\phi_x^* M_{xy}\phi_y + \delta_{xy}(J_x^*\phi_y + \phi_x^* J_y)\right]\right\} \\
&= \exp\left\{-i\int d^4xd^4y J_x^* M_{xy}^{-1} J_y\right\}\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y[(\phi_x^* + J_x^* M_{xy}^{-1})M_{xy}(\phi_y + M_{xy}^{-1}J_y)]\right\},
\end{aligned} \tag{14.1}$$

where, by definition, $M_{xy}^{-1}M_{xy} = M_{xy}M_{xy}^{-1} = \delta_{xy}$. After redefining the fields to absorb the linear shift, which doesn't affect the integral measure as the path integral is supposed to be over the whole field space, we get

$$\begin{aligned}
&\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y[\phi_x^* M_{xy}\phi_y] + i\int d^4x[J_x^*\phi_x + \phi_x^* J_x]\right\} \\
&= \exp\left\{-i\int d^4xd^4y J_x^* M_{xy}^{-1} J_y\right\}\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y \phi_x^* M_{xy}\phi_y\right\} \\
&= \exp\left\{-i\int d^4xd^4y J_x^* M_{xy}^{-1} J_y\right\}\int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left\{i\int d^4xd^4y(\phi_1(x) - i\phi_2(x))M_{xy}(\phi_1(y) + i\phi_2(y))\right\},
\end{aligned} \tag{14.2}$$

where we write out the the two real degrees of freedom of the complex fields.

$$\begin{aligned}
&\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\left\{i\int d^4xd^4y[\phi_x^* M_{xy}\phi_y] + i\int d^4x[J_x^*\phi_x + \phi_x^* J_x]\right\} \\
&= \exp\left\{-i\int d^4xd^4y J_x^* M_{xy}^{-1} J_y\right\} \\
&\times \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left\{i\int d^4xd^4y\left[\phi_1(x)M_{xy}\phi_1(y) + \phi_2(x)M_{xy}\phi_2(y) + i(\phi_1(x)M_{xy}\phi_2(y) - \phi_2(x)M_{xy}\phi_1(y))\right]\right\}
\end{aligned} \tag{14.3}$$

Now notice that the fields are really just classical fields inside the path integral. The variables x and y are really dummy and symmetric if we switch them. Also, both ϕ_1 and ϕ_2 run through the whole real field space. Therefore, the contributions from the $\int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\{-\int d^4xd^4y\phi_1(x)M_{xy}\phi_2(y)\}$ and $\int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\{\int d^4xd^4y\phi_2(x)M_{xy}\phi_1(y)\}$ must cancel each other. Also, $\int \mathcal{D}\phi_1 \exp\{i\int d^4xd^4y\phi_1(x)M_{xy}$ and $\int \mathcal{D}\phi_2 \exp\{i\int d^4xd^4y\phi_2(x)M_{xy}\phi_2(y)\}$ are actually the same. Thus,

$$\begin{aligned}
&\int \mathcal{D}\phi^*\mathcal{D}\phi \exp\Big\{i\int d^4xd^4y[\phi_x^* M_{xy}\phi_y] + i\int d^4x[J_x^*\phi_x + \phi_x^* J_x]\Big\} \\
&= \exp\Big\{-i\int d^4xd^4y J_x^* M_{xy}^{-1} J_y\Big\}\left(\int \mathcal{D}\phi_1 \exp\Big\{i\int d^4xd^4y\phi_1(x)M_{xy}\phi_1(y)\Big\}\right)^2 \\
&= \mathcal{N}\frac{1}{\det M}\exp\Big\{-i\int d^4xd^4y J^*(x)M^{-1}(x,y)J(y)\Big\}
\end{aligned} \tag{14.4}$$

14.2

(a) In scalar QED, the interaction Lagrangian is

$$\mathcal{L}_{int} = -ieA_\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] + e^2A_\mu^2|\phi|^2. \tag{14.5}$$

As the charge-conjugation C only swaps ϕ and ϕ^*, clearly, a transformation to the photon field $A_\mu \to -A_\mu$ keeps the photon field's kinetic term and the above interaction term invariant.

(b)

$$\begin{aligned}
\langle\Omega| T\{A_{\mu_1}(q_1)...A_{\mu_n}(q_n)\}|\Omega\rangle &= \frac{1}{Z[0]}\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x\mathcal{L}[A,\phi_i]}A_{\mu_1}(q_1)...A_{\mu_n}(q_n) \\
&= \frac{1}{Z[0]}\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x\mathcal{L}[A,\phi_i]}(-1)^n A_{\mu_1}(q_1)...A_{\mu_n}(q_n) \\
&= \frac{-1}{Z[0]}\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x\mathcal{L}[A,\phi_i]}A_{\mu_1}(q_1)...A_{\mu_n}(q_n) \\
&= -\langle\Omega| T\{A_{\mu_1}(q_1)...A_{\mu_n}(q_n)\}|\Omega\rangle,
\end{aligned} \tag{14.6}$$

where we applied the charge conjugation on the second line and used the fact that n is odd on the third line. Since the charge conjugation only swaps ϕ and ϕ^* and gives a minus sign to each A_μ, the integral measure and $Z[0]$ is left unchanged by these transformation. Thus, $\langle\Omega| T\{A_{\mu_1}(q_1)...A_{\mu_n}(q_n)\}|\Omega\rangle = 0$.

(c) The above derivation never used any conditions on equation of motion. Therefore, Furry's theorem must hold even if the photons are off-shell.

(d) Assuming Weyl basis, when acting on spinors, the charge conjugation does

$$\begin{aligned}
\psi &\xrightarrow{C} -i\gamma_2\psi^*, \\
\psi^* &\xrightarrow{C} -i\gamma_2\psi.
\end{aligned} \tag{14.7}$$

However, since $(-i\gamma_2)(-i\gamma_2) = (-i)^2(\gamma_2)^2 = (-1)(-1) = 1$, the integral measure and the Lagrangian are still left invariant. The derivation then is just follows that in (b). Thus, Furry's theorem also holds in QED.

(e) The above derivation relies on the integral measure and the Lagrangian to be invariant under the charge conjugation transformation. The Furry's theorem does not in general holds in the Standard model, especially considering loops involving gauge boson of weak interaction. However, any fermionic diagrams or sub-diagrams with odd number of photons attached still vanishe following Furry's theorem, as these are just pure QED diagrams.

14.3

(a) We have

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}), \tag{14.8}$$

$$\hat{\pi}(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} (a_p e^{i\vec{p}\vec{x}} - a_p^\dagger e^{-i\vec{p}\vec{x}}). \tag{14.9}$$

Clearly,

$$\frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\phi}(\vec{x}) + i\hat{\pi}(\vec{x})) = \int \frac{d^3p}{(2\pi)^3} a_p e^{i\vec{p}\vec{x}}, \tag{14.10}$$

and

$$\frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\phi}(\vec{x}) - i\hat{\pi}(\vec{x})) = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger e^{-i\vec{p}\vec{x}}. \tag{14.11}$$

With an inverse Fourier transform,

$$a_p = \int d^3x \frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\phi}(\vec{x}) + i\hat{\pi}(\vec{x})) e^{-i\vec{p}\vec{x}}, \tag{14.12}$$

$$a_p^\dagger = \int d^3x \frac{1}{\sqrt{2\omega_p}} (\omega_p \hat{\phi}(\vec{x}) - i\hat{\pi}(\vec{x})) e^{i\vec{p}\vec{x}}. \tag{14.13}$$

(b) Since,

$$\begin{aligned} |\Phi\rangle &= \int \mathcal{D}\Pi \, |\Pi\rangle \langle \Pi|\Phi\rangle \\ &= \int \mathcal{D}\Pi e^{-i\int d^3x \Pi(\vec{x})\Phi(\vec{x})} \, |\Pi\rangle \, . \end{aligned} \tag{14.14}$$

such that,

$$\begin{aligned}
\hat{\pi}(\vec{x})\,|\Phi\rangle &= \int \mathcal{D}\Pi e^{-i\int d^3x \Pi(\vec{x})\Phi(\vec{x})}\hat{\pi}(\vec{x})\,|\Pi\rangle \\
&= \int \mathcal{D}\Pi e^{-i\int d^3x \Pi(\vec{x})\Phi(\vec{x})}\Pi(\vec{x})\,|\Pi\rangle \\
&= \int \mathcal{D}\Pi\mathcal{D}\Phi' e^{-i\int d^3x \Pi(\vec{x})\Phi(\vec{x})}\Pi(\vec{x})\,|\Phi'\rangle\langle\Phi'|\Pi\rangle \\
&= \int \mathcal{D}\Pi\mathcal{D}\Phi' e^{-i\int d^3x \Pi(\vec{x})[\Phi(\vec{x})-\Phi'(\vec{x})]}\Pi(\vec{x})\,|\Phi'\rangle \\
&= -i\int \mathcal{D}\Phi' \frac{\delta}{\delta\Phi'}\Big(\int \mathcal{D}\Pi e^{-i\int d^3x \Pi(\vec{x})[\Phi(\vec{x})-\Phi'(\vec{x})]}\,|\Phi'\rangle\Big) \\
&= -i\int \mathcal{D}\Phi' \frac{\delta}{\delta\Phi'}\big(|\Phi'\rangle\langle\Phi'|\Phi\rangle\big) \\
&= -i\frac{\delta}{\delta\Phi}\,|\Phi\rangle
\end{aligned} \tag{14.15}$$

The procedures are just the field version of expressing the momentum operators in position eigenspace in quantum mechanics.

(c) Just plugging the Eq. (14.12) into

$$\begin{aligned}
0 &= \langle\Phi|\,a_p\,|0\rangle \\
&= \langle\Phi|\int d^3x \frac{1}{\sqrt{2\omega_p}}(\omega_p\hat{\phi}(\vec{x}) + i\hat{\pi}(\vec{x}))e^{-i\vec{p}\vec{x}}\,|0\rangle \\
&= \int d^3x \frac{1}{\sqrt{2\omega_p}}(\omega_p\Phi(\vec{x}) + \frac{\delta}{\delta\Phi(\vec{x})})e^{-i\vec{p}\vec{x}}\,\langle\Phi|0\rangle\,.
\end{aligned} \tag{14.16}$$

(d) Just plugging the Eq. (14.65) and Eq. (14.66) of the book into the above differential equation. We can check explicitly

$$\begin{aligned}
\int d^3x \frac{\delta}{\delta\Phi(\vec{x})}e^{-i\vec{p}\vec{x}}\,\langle\Phi|0\rangle &= \mathcal{N}\int d^3x e^{-i\vec{p}\vec{x}}\frac{\delta}{\delta\Phi(\vec{x})}e^{-\frac{1}{2}\int d^3y d^3z \mathcal{E}(\vec{z},\vec{y})\Phi(\vec{y})\Phi(\vec{z})} \\
&= -\mathcal{N}\int d^3x e^{-i\vec{p}\vec{x}}\int d^3y' \mathcal{E}(\vec{x},\vec{y}')\Phi(\vec{y}')e^{-\frac{1}{2}\int d^3y d^3z \mathcal{E}(\vec{z},\vec{y})\Phi(\vec{y})\Phi(\vec{z})} \\
&= -\mathcal{N}\int d^3x \int d^3y' \int \frac{d^3q}{(2\pi)^3}e^{i\vec{x}(\vec{q}-\vec{p})}e^{-i\vec{q}\vec{y}'}\omega_q\Phi(\vec{y}')e^{-\frac{1}{2}\int d^3y d^3z \mathcal{E}(\vec{z},\vec{y})\Phi(\vec{y})\Phi(\vec{z})} \\
&= -\mathcal{N}\int d^3y' \int \frac{d^3q}{(2\pi)^3}\delta^3(\vec{q}-\vec{p})e^{-i\vec{q}\vec{y}'}\omega_q\Phi(\vec{y}')e^{-\frac{1}{2}\int d^3y d^3z \mathcal{E}(\vec{z},\vec{y})\Phi(\vec{y})\Phi(\vec{z})} \\
&= -\mathcal{N}\int d^3y' e^{-i\vec{p}\vec{y}'}\omega_p\Phi(\vec{y}')e^{-\frac{1}{2}\int d^3y d^3z \mathcal{E}(\vec{z},\vec{y})\Phi(\vec{y})\Phi(\vec{z})} \\
&= -\int d^3y' \omega_p\Phi(\vec{y}')e^{-i\vec{p}\vec{y}'}\,\langle\Phi|0\rangle\,.
\end{aligned} \tag{14.17}$$

The y' is just a dummy integration variable. Clearly, this satisfies $\int d^3x(\omega_p\Phi(\vec{x})+\frac{\delta}{\delta\Phi(\vec{x})})e^{-i\vec{p}\vec{x}}\,\langle\Phi|0\rangle = 0$ and the Eq. (14.65) of the book is indeed the solution of the differential equation.

(e)

$$\begin{aligned}\mathcal{E}(\vec{x},\vec{y}) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})}\omega_p \\ &= \frac{1}{(2\pi)^2}\int_{-1}^{1} d(\cos\theta)\int_0^\infty dpp^2\sqrt{p^2+m^2}e^{ipr\cos\theta} \\ &= \frac{1}{2\pi^2 r}\int_0^\infty dpp\sqrt{p^2+m^2}\frac{e^{ipr}-e^{-ipr}}{2i} \\ &= \frac{1}{2\pi^2 r}\int_0^\infty dpp\sqrt{p^2+m^2}\sin(pr) \\ &= -\frac{1}{2\pi^2 r}\frac{\partial}{\partial r}\int_0^\infty dp\sqrt{p^2+m^2}\cos(pr),\end{aligned} \tag{14.18}$$

where $r = |\vec{x}-\vec{y}|$. Now, the integral can be expressed as a modified Bessel function of the second kind

$$\int_0^\infty dp\sqrt{p^2+m^2}\cos(pr) = -\frac{m}{r}\mathcal{K}_{-1}(mr). \tag{14.19}$$

Thus,

$$\mathcal{E}(\vec{x},\vec{y}) = \frac{m}{2\pi^2 r}\frac{\partial}{\partial r}(\frac{1}{r}\mathcal{K}_{-1}(mr)) \tag{14.20}$$

For m = 0,

$$\begin{aligned}\mathcal{E}(\vec{x},\vec{y}) &= -\frac{1}{2\pi^2 r}\frac{\partial}{\partial r}\int_0^\infty dpp\cos(pr) \\ &= -\frac{1}{2\pi^2 r}\frac{\partial^2}{\partial r^2}\int_0^\infty dp\sin(pr) \\ &= -\frac{1}{2\pi^2 r}\frac{\partial^2}{\partial r^2}(\frac{1}{r}) \\ &= -\frac{1}{\pi^2 r^4}\end{aligned} \tag{14.21}$$

14.4

(a) Expanding the $f_z(a^\dagger)$ as a power series of $a^\dagger$, $f_z(a^\dagger) = a_0 + a_1 a^\dagger + a_2(a^\dagger)^2 + \ldots$, where a_n are coefficients. Then, notice

$$\begin{aligned}af_z(a^\dagger)|0\rangle &= a(a_0 + a_1 a^\dagger + a_2(a^\dagger)^2 + \ldots)|0\rangle \\ &= (a_1(1+a^\dagger a) + a_2(1+a^\dagger a)a^\dagger + \ldots)|0\rangle \\ &= (a_1 + 2a_2 a^\dagger + 3a_3(a^\dagger)^2 + \ldots)|0\rangle \\ &= \frac{d(f_z(a^\dagger))}{da^\dagger}|0\rangle\end{aligned} \tag{14.22}$$

Thus,

$$\begin{aligned} \hat{x}\left|\psi\right\rangle &= z\left|\psi\right\rangle \\ c(a+a^\dagger)f_z(a^\dagger)\left|0\right\rangle &= zf_z(a^\dagger)\left|0\right\rangle \\ (\frac{d(f_z(a^\dagger))}{da^\dagger}+a^\dagger f_z(a^\dagger))\left|0\right\rangle &= \frac{z}{c}f_z(a^\dagger)\left|0\right\rangle . \end{aligned} \tag{14.23}$$

We thus have a differential equation $\frac{d(f_z)}{da^\dagger}+a^\dagger f_z = \frac{z}{c}f_z$, which has solution $f_z(a^\dagger) = Ne^{-\frac{1}{2}(a^\dagger)^2+\frac{z}{c}a^\dagger}$, where N is some normalization constant.

To fix N, we can notice that

$$\langle 0|\psi\rangle = \left\langle 0\right| f_z \left|0\right\rangle = N. \tag{14.24}$$

To solve for $\langle 0|\psi\rangle$, we can use the differential equation Eq. (14.16) in problem 14.3. The QM version of the differential equation is just

$$(\omega z + \frac{d}{dx})\left\langle \psi|0\right\rangle = 0, \tag{14.25}$$

which has solution $\psi_0 = \langle\psi|0\rangle = N'e^{-\frac{1}{2}\omega z^2}$, which is the famous ground state wavefunction of a harmonic oscillator system. Requiring the ground state to be properly normalized

$$1 = \int dz\psi_0\psi_0^* = |N'|^2\int dz e^{-\omega x^2} = |N'|^2\sqrt{\frac{\pi}{\omega}} \tag{14.26}$$

Thus,

$$\begin{aligned} f_z(a^\dagger) &= (\frac{\omega}{\pi})^{\frac{1}{4}}e^{-\frac{1}{2}(a^\dagger)^2+\sqrt{2\omega}za^\dagger-\frac{1}{2}\omega z^2} \\ &= (\frac{\omega}{\pi})^{\frac{1}{4}}e^{-\frac{1}{2}(a^\dagger-\sqrt{2\omega}z)^2+\frac{\omega z^2}{2}}, \end{aligned} \tag{14.27}$$

where we replaced $c=\frac{1}{\sqrt{2\omega}}$ into the expression.

(b) To generalize the above construction to field theory, we shall do the following replacement

$$\begin{aligned} a_p^\dagger \to \hat{\phi}^+(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p^\dagger e^{-i\vec{p}\vec{x}}, \\ a_p \to \hat{\phi}^-(x) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p e^{i\vec{p}\vec{x}}, \\ \hat{x} \to \hat{\phi}(x) &= \hat{\phi}^+(x)+\hat{\phi}^-(x). \end{aligned} \tag{14.28}$$

And

$$\left[\hat{\phi}^\pm(x),\hat{\phi}^\pm(y)\right] = 0, \tag{14.29}$$

$$D_{xy} \equiv \left[\hat{\phi}^-(x),\hat{\phi}^+(y)\right] = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{i\vec{p}(\vec{x}-\vec{y})}, \tag{14.30}$$

$$\hat{\phi}^-(y)\left|0\right\rangle = 0, \tag{14.31}$$

where we used lower indices to denote coordinates.

Define $f(\hat{\phi}^+)$ as $f(\hat{\phi}^+)\left|0\right\rangle = \left|\Phi\right\rangle$, where $\left|\Phi\right\rangle$ is the eigenstate of $\hat{\phi}$ such that $\hat{\phi}(x)\left|\Phi\right\rangle = \Phi(x)\left|\Phi\right\rangle$. Expanding $f(\hat{\phi}^+)$ as

$$f(\hat{\phi}^+) = a_0 + a_1 \int dx \phi^+(y) + a_2 \int dx \int dy \phi^+(x)\phi^+(y) + ... \tag{14.32}$$

With the same procedures as (a), we should arrive

$$\hat{\phi}_x^- f(\hat{\phi}^+) = D_{xy} \frac{d(f(\hat{\phi}^+))}{d\hat{\phi}_y^+}. \tag{14.33}$$

Notice repeated indices are being integrated over here and in the following. Then,

$$\begin{aligned} \hat{\phi}_x \left|\Phi\right\rangle &= (\hat{\phi}_x^+ + \hat{\phi}_x^-) f(\hat{\phi}^+)\left|0\right\rangle = \Phi_x f(\hat{\phi}^+)\left|0\right\rangle \\ D_{xy}\frac{d(f(\hat{\phi}^+))}{d\hat{\phi}_y^+} &+ \hat{\phi}_x^+ f(\hat{\phi}^+) = \Phi_x f(\hat{\phi}^+) \end{aligned} \tag{14.34}$$

The solution to the differential equation is given by

$$f(\hat{\phi}^+) = \mathcal{N} \exp\left\{ -(\hat{\phi}_x^+ - \Phi_x)\mathcal{E}_{xy}(\hat{\phi}_y^+ - \Phi_y) + \frac{1}{2}\Phi_x \mathcal{E}_{xy}\Phi_y \right\}, \tag{14.35}$$

where $\mathcal{E}_{xy}$ is given by Eq. (14.66) of the book. It's easy to check that this solution satisfies the above differential equation and also has the correct boundary condition.

$$\begin{aligned} D_{xy}\frac{d(f(\hat{\phi}^+))}{d\hat{\phi}_y^+} &= -D_{xy}[\delta_{x'y}\mathcal{E}_{x'y'}(\hat{\phi}_{y'}^+ - \Phi_{y'}) + \delta_{y'y}(\hat{\phi}_{x'}^+ - \Phi_{x'})\mathcal{E}_{x'y'}]f(\hat{\phi}^+) \\ &= -[D_{xy}\mathcal{E}_{yy'}(\hat{\phi}_{y'}^+ - \Phi_{y'}) + D_{xy}(\hat{\phi}_{x'}^+ - \Phi_{x'})\mathcal{E}_{x'y}]f(\hat{\phi}^+). \end{aligned} \tag{14.36}$$

Since

$$\begin{aligned} D_{xy}\mathcal{E}_{yy'} &= \int dy \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{\omega_q}{2\omega_p} e^{i\vec{p}\vec{x} - i\vec{q}\vec{y}'} e^{i\vec{y}(\vec{q}-\vec{p})} \\ &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{\omega_q}{2\omega_p} e^{i\vec{p}\vec{x} - i\vec{q}\vec{y}'} (2\pi)^3 \delta^3(\vec{q}-\vec{p}) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y}')} \\ &= \frac{1}{2}\delta^3_{xy'}, \end{aligned} \tag{14.37}$$

$$\begin{aligned} D_{xy}\frac{d(f(\hat{\phi}^+))}{d\hat{\phi}_y^+} &= -[\frac{1}{2}\delta^3_{xy'}(\hat{\phi}_{y'}^+ - \Phi_{y'}) + \frac{1}{2}\delta^3_{xx'}(\hat{\phi}_{x'}^+ - \Phi_{x'})]f(\hat{\phi}^+) \\ &= -(\hat{\phi}_x^+ - \Phi_x)f(\hat{\phi}^+), \end{aligned} \tag{14.38}$$

which is clearly Eq. (14.34). Also $\left|\Phi\right\rangle = \mathcal{N}\exp\left\{ -(\hat{\phi}_x^+ - \Phi_x)\mathcal{E}_{xy}(\hat{\phi}_y^+ - \Phi_y) + \frac{1}{2}\Phi_x\mathcal{E}_{xy}\Phi_y \right\}\left|0\right\rangle$ has the correct boundary condition.

$$\left\langle 0|\Phi\right\rangle = \mathcal{N}\exp\left\{ -\frac{1}{2}\Phi_x\mathcal{E}_{xy}\Phi_y \right\}, \tag{14.39}$$

which is exactly the Eq. (14.65) of the book and we have already shown in problem 14.3 that it is the right boundary condition.

(c) Similarly, we can construct the eigenstate of $\hat{\pi} = \hat{\pi}^+ + \hat{\pi}^-$, where

$$\hat{\pi}^+ = i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}a_p^\dagger e^{-i\vec{p}\vec{x}}, \hat{\pi}^- = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}a_p e^{i\vec{p}\vec{x}}. \tag{14.40}$$

$$\left[\hat{\pi}^-(x), \hat{\pi}^+(y)\right] = \int \frac{d^3p}{(2\pi)^3}\frac{\omega_p}{2}e^{i\vec{p}(\vec{x}-\vec{y})} = \frac{1}{2}\mathcal{E}_{xy}. \tag{14.41}$$

So,

$$|\Pi\rangle = \mathcal{N}\exp\{-2(\hat{\pi}_x^+ - \Pi_x)D_{xy}(\hat{\pi}_y^+ - \Pi_y) + \Pi_x D_{xy}\Pi_y\}|0\rangle\,. \tag{14.42}$$

Also,

$$\langle 0|\,\hat{\pi}_x^-\hat{\phi}_y^+\,|0\rangle = -\frac{i}{2}\delta_{xy} = -\left[\hat{\pi}_x^-, \hat{\phi}_y^+\right]. \tag{14.43}$$

Also,

$$\begin{aligned}\Big[-2(\hat{\pi}_{x'}^- - \Pi_{x'})D_{x'y'}(\hat{\pi}_{y'}^- - \Pi_y), -(\hat{\phi}_x^+ - \Phi_x)\mathcal{E}_{xy}(\hat{\phi}_y^+ - \Phi_y)\Big] &= 2D_{x'y'}\mathcal{E}_{xy}(\Big[\hat{\pi}_{x'}^-\hat{\pi}_{y'}^-, \hat{\phi}_x^+\hat{\phi}_y^+\Big] \\ &\quad + \Big[\Pi_{y'}\hat{\pi}_{x'}^- + \Pi_{x'}\hat{\pi}_{y'}^-, \hat{\phi}_x^+\Phi_y + \Phi_x\hat{\phi}_y^+\Big]) \\ &\quad + ...,\end{aligned} \tag{14.44}$$

where (...) contains terms that have unequal number of $\hat{\pi}$ and $\hat{\phi}^+$, which when sandwiched by $|0\rangle$ vanish. Also notice the first term $\Big[\hat{\pi}_{x'}^-\hat{\pi}_{y'}^-, \hat{\phi}_x^+\hat{\phi}_y^+\Big]$ just produced infinite c-number that can be absorbed into the normalization constant $\mathcal{N}$. Thus, the only terms that we need to calculate is just

$$\begin{aligned}&2D_{x'y'}\mathcal{E}_{xy}\Big[\Pi_{y'}\hat{\pi}_{x'}^- + \Pi_{x'}\hat{\pi}_{y'}^-, \hat{\phi}_x^+\Phi_y + \Phi_x\hat{\phi}_y^+\Big]) \\ &\quad = 2D_{x'y'}\mathcal{E}_{xy}(\Pi_{y'}\Phi_y\Big[\hat{\pi}_{x'}^-, \hat{\phi}_x^+\Big] + \Pi_{y'}\Phi_x\Big[\hat{\pi}_{x'}^-, \hat{\phi}_y^+\Big] + \Pi_{x'}\Phi_y\Big[\hat{\pi}_{y'}^-, \hat{\phi}_x^+\Big] + \Pi_{x'}\Phi_x\Big[\hat{\pi}_{y'}^-, \hat{\phi}_y^+\Big]) \\ &\quad = iD_{x'y'}\mathcal{E}_{xy}(\Pi_{y'}\Phi_y\delta_{x'x} + \Pi_{y'}\Phi_x\delta_{x'y} + \Pi_{x'}\Phi_y\delta_{y'x} + \Pi_{x'}\Phi_x\delta_{y'y}) \\ &\quad = iD_{xy'}\mathcal{E}_{xy}\Pi_{y'}\Phi_y + iD_{yy'}\mathcal{E}_{xy}\Pi_{y'}\Phi_x + iD_{x'x}\mathcal{E}_{xy}\Pi_{x'}\Phi_y + iD_{x'y}\mathcal{E}_{xy}\Pi_{x'}\Phi_x \\ &\quad = \frac{i}{2}(\delta_{y'y}\Pi_{y'}\Phi_y + \delta_{y'x}\Pi_{y'}\Phi_x + \delta_{x'y}\Pi_{x'}\Phi_y + \delta_{x'x}\Pi_{x'}\Phi_x) \\ &\quad = 2i\Pi_x\Phi_x\end{aligned} \tag{14.45}$$

Then, using the identities $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

$$\langle\Pi|\Phi\rangle = \mathcal{N}\,\langle 0|0\rangle\exp\{-i\Pi_x\Phi_x\} = \exp\{-i\Pi_x\Phi_x\}, \tag{14.46}$$

where $\langle 0|0\rangle$ is properly normalized with respect to $\mathcal{N}$. This is exactly the Eq. (14.21) of the book. Eq. (14.22) thus follows.

14.5

(a) The Lagrangian for scalar QED (cf. Eq. (9.11) of the book) is

$$\mathcal{L} = \frac{1}{2}A_\mu\Box^{\mu\nu}A_\nu - \phi^*(\Box + m^2)\phi - ieA_\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] + e^2A_\mu^2|\phi|^2. \tag{14.47}$$

Under a field redefinition, $A_\mu(x) \to A_\mu(x) + \varepsilon_\mu(x)$,

$$\mathcal{L} \to \mathcal{L} + \varepsilon_\mu(x)\left\{\Box_{\mu\nu}A_\nu - ie\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] + 2e^2A_\mu|\phi|^2\right\}, \tag{14.48}$$

where we only retain terms to first order in ε_μ. Now considering the correlation function $\langle A^\alpha\phi^*\phi\rangle$, we would find

$$\begin{aligned}\langle A_\alpha\phi^*\phi\rangle &= \frac{1}{Z[0]}\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\alpha e^{i\int d^4x\left[\mathcal{L}+\varepsilon_\mu(x)\left\{\Box_{\mu\nu}A_\nu - ie\left[\phi^*(\partial_\mu\phi)-(\partial_\mu\phi^*)\phi\right]+2e^2A_\mu|\phi|^2\right\}\right)]}\\ &\qquad\qquad \times\left[A_\alpha(x_1)+\varepsilon_\alpha(x_1)\right]\phi^*(x_2)\phi(x_3)\\ \Longrightarrow\ 0 &= \int d^4x\varepsilon_\mu(x)\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\alpha e^{iS}\{\Box^x_{\mu\nu}A_\nu(x)A_\alpha(x_1)\\ &\qquad\qquad + \left[-ie(\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi) + 2e^2A_\mu|\phi|^2\right]A_\alpha(x_1)\\ &\qquad\qquad + i\delta^4(x-x_1)g_{\mu\alpha}\}\phi^*(x_2)\phi(x_3).\end{aligned} \tag{14.49}$$

This gives the Schwinger-Dyson equation

$$\begin{aligned}&\Box^x_{\mu\nu}\langle A_\nu(x)A_\alpha(x_1)\phi^*(x_2)\phi(x_3)\rangle =\\ &\qquad ie\langle\phi^*(x)(\partial_\mu\phi(x))A_\alpha(x_1)\phi^*(x_2)\phi(x_3)\rangle - ie\langle(\partial_\mu\phi^*(x))\phi(x)A_\alpha(x_1)\phi^*(x_2)\phi(x_3)\rangle\\ &\qquad - 2e^2\langle A_\mu(x)|\phi(x)|^2A_\alpha(x_1)\phi^*(x_2)\phi(x_3)\rangle - i\delta^4(x-x_1)g_{\mu\alpha}\langle\phi^*(x_2)\phi(x_3)\rangle\\ &\quad = e\langle j_\mu(x)A_\alpha(x_1)\phi^*(x_2)\phi(x_3)\rangle - i\delta^4(x-x_1)g_{\mu\alpha}\langle\phi^*(x_2)\phi(x_3)\rangle,\end{aligned} \tag{14.50}$$

where now $j_\mu = -i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - 2eA_\mu\phi^*\phi$.

(b) We should consider the correlation function $\langle\phi^*(x_1)\phi(x_2)\rangle$. With a field redefinition of $\phi(x) \to e^{-i\alpha(x)}\phi(x)$ and $\phi^*(x) \to e^{i\alpha(x)}\phi^*(x)$, the free ($e = 0$) scalar QED Lagrangian transform as

$$\mathcal{L}_0 \to \mathcal{L}_0 + i(\partial_\mu\alpha)(\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi) + (\partial_\mu\alpha)^2|\phi|^2, \tag{14.51}$$

while

$$\phi^*(x_1)\phi(x_2) \to e^{i\alpha(x_1)}e^{-i\alpha(x_2)}\phi^*(x_1)\phi(x_2). \tag{14.52}$$

Following the same steps as in the Sec. 14.8.1 of the book, expanding to first order in α, we arrive at

$$\begin{aligned}&\int d^4x\alpha(x)i\partial_\mu\int\mathcal{D}\phi\mathcal{D}\phi^*e^{iS}\left[\phi^*(x)(\partial_\mu\phi(x)) - (\partial_\mu\phi^*(x))\phi(x)\right]\phi^*(x_1)\phi(x_2)\\ &\qquad = \int d^4x\alpha(x)\left[i\delta(x-x_1) - i\delta(x-x_2)\right]\int\mathcal{D}\phi\mathcal{D}\phi^*e^{iS}\phi^*(x_1)\phi(x_2),\end{aligned} \tag{14.53}$$

which implies

$$\partial_\mu \langle j_0^\mu(x)\phi^*(x_1)\phi(x_2)\rangle = \delta(x-x_1)\langle\phi^*(x_1)\phi(x_2)\rangle - \delta(x-x_2)\langle\phi^*(x_1)\phi(x_2)\rangle \tag{14.54}$$

for free theory ($e=0$) and where $j_0^\mu = -i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$. However, when generalizing higher-order correlation functions involving interaction with the photon fields, the situation is quite different from that of spinor QED. While in spinor QED, the only interaction term $A_\mu\bar{\psi}\gamma^\mu\psi$ is invariant under the field redefinition of the spinor fields, in scalar QED, the interaction term $-ieA_\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right]$ part is not invariant under the similar field redefinition to the scalar field. To see the effect in correlation function, we need to use the full scalar QED Lagrangian and observe it transforms as

$$\mathcal{L} \to \mathcal{L} + i(\partial_\mu\alpha)(\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi) + (\partial_\mu\alpha)^2|\phi|^2 - 2e(\partial_\mu\alpha)A_\mu|\phi|^2. \tag{14.55}$$

Then similar procedure leads to

$$\partial_\mu \langle j^\mu(x)\phi^*(x_1)\phi(x_2)\rangle = \delta(x-x_1)\langle\phi^*(x_1)\phi(x_2)\rangle - \delta(x-x_2)\langle\phi^*(x_1)\phi(x_2)\rangle, \tag{14.56}$$

where now $j_\mu = -i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - 2eA_\mu\phi^*\phi$, in consistence with the results in (a). It should be noticed that the scalar QED current in free theory is not gauge-invariant and thus, not physical. Only if one include the photon field, does the current become gauge-invariant and physical.

14.6

(a)

(b)

Part III

Renormalization

Chapter 15

The Casimir effect ✓

15.1

The Gaussian regulator is

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-(\frac{\omega_n}{\pi\Lambda})^2} \tag{15.1}$$

Expanding with $\omega_n = \frac{\pi}{r}n$:

$$E(r) = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{\infty} n e^{-(\frac{n}{r\Lambda})^2} = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{\infty} n e^{-(\epsilon n)^2}, \quad \epsilon = \frac{1}{\Lambda r} \ll 1. \tag{15.2}$$

Now using the Euler-Maclaurin series to calculate the sum:

$$\begin{aligned} \sum_{n=1}^{\infty} n e^{\epsilon^2 n^2} - \int_0^{\infty} n e^{\epsilon^2 n^2} dn &= -\frac{1}{12} + \mathcal{O}(\epsilon^2) \\ \sum_{n=1}^{\infty} n e^{\epsilon^2 n^2} &= \frac{1}{2\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{15.3}$$

Then

$$E(r) = \frac{\pi}{4}r\Lambda^2 - \frac{\pi}{24r} + \mathcal{O}\left(\frac{1}{r^2\Lambda}\right) \tag{15.4}$$

$$\begin{aligned} F(a) &= -\frac{d}{da}[E(L-a) + E(a)] = -\frac{d}{da}\left[\frac{\pi}{4}L\Lambda^2 - \frac{\pi}{24}(\frac{1}{L-a} + \frac{1}{a}) + \dots\right] \\ &= \frac{\pi}{24}(\frac{1}{(L-a)^2} - \frac{1}{a^2}) + \dots \end{aligned} \tag{15.5}$$

Now take $L \to \infty$ and we again get

$$F(a) = -\frac{\pi\hbar c}{24a^2} \tag{15.6}$$

15.2

Simply take a look on (12.67) of the book,

$$E = \sum_s \left[\int \frac{d^3q}{(2\pi)^3}\omega_p(a_p^{s\dagger}a_p^s + b_p^{s\dagger}b_p^s) - V\varepsilon_0\right]. \tag{15.7}$$

It's easy to see that the zero-point energy is negative while for bosons, the zero-point energy is positive. The argument would then follow similarly as the scalar case in the book except an overall sign flipped, which makes the final answer of the Casimir force to have an opposite sign than from bosons.

15.3

$A \sim (0.5\mu m)^2 \approx 2.5 \times 10^{-13} m^2$. Assuming the length of each setate is about $5nm$. Plugging these values into the 3-d Casimir force formula from (15.22) of the book, each setate would provide a force around $F \sim \frac{\pi^2 \hbar c}{240 a^4} A \sim 0.5\mu N$. With a million of setates on each foot and a gecko has four feet, these would provide a force about 2 N, which should be enough to hold a gecko weight up to $200g$ to climb on walls.

15.4

By simple dimensional analysis, since the mass contributes positively with respect to the energy and has inverse dimension with the length of small box a, it's expected that the effect of mass would lead to a term of Casimir force that is opposite to that of a purely massless field. In other words, the mass should introduce a repulsive term in the Casimir force. For a massive scalar fields in d dimensions between two plates separated by a distance a, the energy for its $n-$th mode

$$\omega_k = \sqrt{(\frac{n\pi}{a})^2 + k_T^2 + m^2}, \tag{15.8}$$

where k_T is the momentum for transverse modes with respect to the dimension on where we placed the cavity. Thus, the total ground states energy for a d dimensional box with width L

on each sides is

$$\begin{aligned}
E &= L^{d-1}\int\frac{d^{d-1}(k_T)}{(2\pi)^{d-1}}\sum_{n=1}\frac{1}{2}\omega_k\\
&=(\frac{L}{2\pi})^{d-1}\int d\Omega_{d-1}\int dk_T k_T^{d-2}\sum_{n=1}\frac{1}{2}\omega_k\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{2}\int_0^\infty d(k_T^2)(k_T^2)^{\frac{d-3}{2}}\frac{1}{2}\sum_{n=1}\sqrt{(\frac{n\pi}{a})^2+k_T^2+m^2}\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{2}\int_0^\infty d(k_T^2)(k_T^2)^{\frac{d-3}{2}}\frac{1}{2}\sum_{n=1}\sqrt{(\frac{n\pi}{a})^2+m^2}\sqrt{1+\frac{k_T^2}{(\frac{n\pi}{a})^2+m^2}}\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{4}\sum_{n=1}((\frac{n\pi}{a})^2+m^2)^{\frac{d}{2}}\int_0^\infty d(\frac{k_T^2}{(\frac{n\pi}{a})^2+m^2})(\frac{k_T^2}{(\frac{n\pi}{a})^2+m^2})^{\frac{d-3}{2}}\sqrt{1+\frac{k_T^2}{(\frac{n\pi}{a})^2+m^2}}\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{4}\sum_{n=1}((\frac{n\pi}{a})^2+m^2)^{\frac{d}{2}}\beta(\frac{d-1}{2},-\frac{d}{2})\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{4}\sum_{n=1}((\frac{n\pi}{a})^2+m^2)^{\frac{d}{2}}\frac{\Gamma(\frac{d-1}{2})\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})}\\
&=(\frac{L}{2\pi})^{d-1}\frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}\frac{1}{4}\sum_{n=1}((\frac{n\pi}{a})^2+m^2)^{\frac{d}{2}}\frac{\Gamma(\frac{d-1}{2})\Gamma(-\frac{d}{2})}{\Gamma(-\frac{1}{2})}\\
&=-(\frac{L}{2})^{d-1}\frac{\Gamma(-\frac{d}{2})}{4}\sum_{n=1}((\frac{n}{a})^2\pi+\frac{m^2}{\pi})^{\frac{d}{2}},
\end{aligned}\tag{15.9}$$

where we have used one of the defining integral of β function to do the integral $\beta(a,b)=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt$.

Now it's quite clear that for any positive d, the sum is divergent, so we must find an analytic continuity to positive d and extract the divergent part, so let's start with negative d first,

$$\begin{aligned}
E &= -(\frac{L}{2})^{d-1}\frac{\Gamma(-\frac{d}{2})}{4}\sum_{n=1}^\infty\frac{1}{((\frac{n}{a})^2\pi+\frac{m^2}{\pi})^{-\frac{d}{2}}}\\
&=-(\frac{L}{2})^{d-1}\frac{1}{4}\sum_{n=1}^\infty\int_0^\infty x^{-\frac{d}{2}-1}e^{-((\frac{n}{a})^2\pi+\frac{m^2}{\pi})x}dx.
\end{aligned}\tag{15.10}$$

Now notice that the infinite sum can be turned into a form involving Jacobi theta function and then applied with Jacobi identities

$$\begin{aligned}\psi\left(\frac{x}{a}\right) &= \sum_{n=1}^{\infty} e^{-(\frac{n}{a})^2\pi x} \\ &= \frac{1}{2}[\theta_3(0; e^{-\frac{\pi x}{a^2}}) - 1] \\ &= \frac{1}{2}[\vartheta_{00}(0; \frac{ix}{a^2}) - 1] \\ &= \frac{1}{2}\left[\frac{a}{\sqrt{x}}\vartheta_{00}(0; \frac{ia^2}{x}) - 1\right] \\ &= \frac{a}{\sqrt{x}}(\sum_{n=1}^{\infty} e^{-\frac{n^2\pi a^2}{x}}) + \frac{a}{2\sqrt{x}} - \frac{1}{2}.\end{aligned} \tag{15.11}$$

Put this back,

$$\begin{aligned}E &= -(\frac{L}{2})^{d-1}\frac{1}{4}\left[a\sum_{n=1}^{\infty}(\int_0^{\infty} x^{-\frac{d+3}{2}} e^{-\frac{n^2\pi a^2}{x} - \frac{m^2}{\pi}x} dx) + \frac{a}{2}\int_0^{\infty} x^{-\frac{d+3}{2}} e^{-\frac{m^2}{\pi}x} dx - \frac{1}{2}\int_0^{\infty} x^{-\frac{d}{2}-1} e^{-\frac{m^2}{\pi}x} dx\right] \\ &= -(\frac{L}{2})^{d-1}\frac{1}{4}\left[a(\frac{m^2}{\pi})^{\frac{d+1}{2}}\sum_{n=1}^{\infty}\frac{2\mathcal{K}_{\frac{d+1}{2}}(2man)}{(man)^{\frac{d+1}{2}}} + \frac{a}{2}(\frac{m^2}{\pi})^{\frac{d+1}{2}}\Gamma(-\frac{d+1}{2}) - \frac{1}{2}(\frac{m^2}{\pi})^{\frac{d}{2}}\Gamma(-\frac{d}{2})\right],\end{aligned} \tag{15.12}$$

where $\mathcal{K}$ is the modified Bessel function and we have used $\mathcal{K}_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty \exp\left\{-t - \frac{z^2}{4t}\right\}\frac{dt}{t^{\nu+1}}$. The d in first term can be taken smoothly into positive number, so all the divergent parts due to positive dimensions have been moved into second and third term. For space dimension $d = 3$, the second term has simple pole and is not physical, while the third term has no a dependence and thus is irrelevant to the calculation of Casimir force. Only the first term is relevant,

$$E = -\frac{L^2 m^2}{8\pi^2 a}\sum_{n=1}^{\infty}\frac{\mathcal{K}_2(2man)}{n^2} + ... \tag{15.13}$$

For limiting case $m \ll a^{-1}$, we can expand the modified Bessel function $\mathcal{K}_2(2man) = \frac{1}{2(man)^2} - \frac{1}{2} + \mathcal{O}(m^2)$, so

$$\begin{aligned}E &\approx -\frac{L^2}{16\pi^2 a^3}\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{L^2 m^2}{16\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2} + ... \\ &= -\frac{L^2}{16\pi^2 a^3}\zeta(4) + \frac{L^2 m^2}{16\pi^2 a}\zeta(2) + ... \\ &= -\frac{L^2\pi^2}{1440 a^3} + \frac{L^2 m^2}{96a}....\end{aligned} \tag{15.14}$$

The Casimir force for $m \ll a^{-1}$ limit is then,

$$F(a) = -\frac{dE}{da} = -\frac{L^2\pi^2}{480a^4} + \frac{L^2 m^2}{96a^2}. \tag{15.15}$$

Notice the first term is exactly the result of a massless field as the Eq. (15.22) of the book. The factor of 2 is due to the book has accounted for the two photon polarizations. The second term is how the mass of particle would modify the Casimir force, which is a repulsive force as expected.

For $m \gg a^{-1}$, the asymptotic behaviour of the modified Bessel function is $\mathcal{K}_2(2amn) \to \sqrt{\frac{\pi}{4amn}}e^{-2amn}$. Thus

$$E \to \frac{L^2m^2}{16\pi^2 a}\sqrt{\frac{\pi}{am}}\sum_{n=1}^{\infty}\frac{e^{-2amn}}{n^{\frac{5}{2}}} = \frac{L^2m^2}{16\pi^2 a}\sqrt{\frac{\pi}{am}}Li_{\frac{5}{2}}(e^{-2am}) \to \frac{L^2m^2}{16\pi^2 a}\sqrt{\frac{\pi}{am}}e^{-2am}, \tag{15.16}$$

where Li is the polylogarithm and has the limiting behaviour $\lim_{|z|\to 0} Li_s(z) = z$.

The Casimir energy and so is the Casimir force at this limit is exponentially small (but the force is clearly still repulsive), which corresponds to the classical limit when the mass is very large.

Chapter 16

Vacuum polarization

16.1

- **Scalar Case:**

  Starting from (16.24) of the book,

$$i\Pi_2^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2p^\mu k^\nu + 2p^\nu k^\mu - p^\mu p^\nu + 2g^{\mu\nu}[(p-k)^2 - m^2]}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \tag{16.1}$$

  After manipulating with the Feynman parameters and doing the shift $k^\mu \to k^\mu + p^\mu(1-x)$, the terms that can potentially contribute to the $p^\mu p^\nu$ are

$$\begin{aligned}
\Pi_2^{\mu\nu} &= ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4p^\mu p^\nu(1-x)^2 + 4p^\mu p^\nu(1-x) - p^\mu p^\nu}{[k^2 + p^2x(1-x) - m^2 + i\varepsilon]^2} + g^{\mu\nu} term \\
&= ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{p^\mu p^\nu(-4x^2 + 4x - 1)}{[k^2 + p^2x(1-x) - m^2 + i\varepsilon]^2} + g^{\mu\nu} term \\
&= -2\frac{e^2}{(4\pi)^{d/2}} p^\mu p^\nu \Gamma(2 - \frac{d}{2})\mu^{4-d} \int_0^1 dx(-2x^2 + 2x - \frac{1}{2})(\frac{1}{\Delta})^{2-\frac{d}{2}} + g^{\mu\nu} term \\
&= -2\frac{e^2}{(4\pi)^{d/2}} p^\mu p^\nu \Gamma(2 - \frac{d}{2})\mu^{4-d} \int_0^1 dx \left[-x(2x-1) + (x - \frac{1}{2})\right] (\frac{1}{\Delta})^{2-\frac{d}{2}} + g^{\mu\nu} term,
\end{aligned} \tag{16.2}$$

  with $\Delta = m^2 - p^2x(1-x)$. Now notice that if we do a linear shift $x \to x + \frac{1}{2}$ for the second term under the integral,

$$\begin{aligned}
\int_0^1 dx(x - \frac{1}{2})(\frac{1}{\Delta})^{2-\frac{d}{2}} &= \int_{-1/2}^{1/2} x \left(\frac{1}{m^2 - p^2(x + \frac{1}{2})(\frac{1}{2} - x)}\right) \\
&= 0,
\end{aligned} \tag{16.3}$$

  which vanishes due to oddness. Thus we have

$$\Pi_2^{\mu\nu} = -2\frac{e^2}{(4\pi)^{d/2}} p^\mu p^\nu \Gamma\left(2 - \frac{d}{2}\right) \mu^{4-d} \int_0^1 dx[-x(2x-1)] \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} + g^{\mu\nu} term, \tag{16.4}$$

  Compared with (16.38) of the book, this is consistent with the Ward identity.

- **Spinor Case:**

  Starting from Eq. (16.42) of the book. We can see the numerator terms that could contribute to the $p^\mu p^\nu$ are

$$N = 4[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu] \tag{16.5}$$

After the shifting $k^\mu \to k^\mu + p^\mu(1-x)$, then dropping the linear p^μ and p^ν terms as they are odd under $k \to -k$, the non-zero terms still contributing to the $p^\mu p^\nu$ are

$$N = 8p^\mu p^\nu(1-x)^2 - 8p^\mu p^\nu(1-x) = -8p^\mu p^\nu x(1-x) \tag{16.6}$$

Then,

$$\begin{aligned}\Pi_2^{\mu\nu} &= 8ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-p^\mu p^\nu x(1-x)}{[k^2 + p^2x(1-x) - m^2 + i\varepsilon]^2} + g^{\mu\nu} term \\ &= 8\frac{e^2}{(4\pi)^{d/2}} p^\mu p^\nu \Gamma(2-\frac{d}{2})\mu^{4-d} \int_0^1 dx(1-x)x(\frac{1}{\Delta})^{2-\frac{d}{2}} + g^{\mu\nu} term \\ &= \frac{-8e^2}{(4\pi)^{d/2}}(p^2 g^{\mu\nu} - p^\mu p^\nu)\Gamma(2-\frac{d}{2})\mu^{4-d} \int_0^1 dx(1-x)x(\frac{1}{\Delta})^{2-\frac{d}{2}},\end{aligned} \tag{16.7}$$

where $\delta = m^2 - p^2x(1-x)$. Again, the result is consistent with Ward identity.

16.2

Notice that the momentum space potential Eq. (16.56) is rotationally invariant, so we can take the results from the Eq. (3.64) of the book, taking the Born approximation:

$$V(r) = \frac{ie^2}{8\pi^2 r} \int_{-\infty}^{\infty} dp \frac{1 - e^2[\Pi_2(p^2) - \Pi_2(0)]}{p + i\varepsilon} e^{-ipr}. \tag{16.8}$$

One should then close the contour down to perform the integral. For the leading order, this just gives the usual Coulomb potential

$$V(r) = \frac{ie^2}{8\pi^2 r}(-2\pi i)(-e^{-\varepsilon r}) = -\frac{e^2}{4\pi r}. \tag{16.9}$$

Now for the correction term, there is also a contribution from the branch cut of the log function. Due to the prefactor if the i in front of the integral and the fact that the potential, as an observable must be real, the contribution can only come from the imaginary part of $\Pi_2(p^2) - \Pi_2(0)$ (this is actually a result of spectral representation. See Sec. 24.2 of the book). From the the Eq. (16.55) of the book,

$$I(p^2) \equiv \Pi_2(p^2) - \Pi_2(0) = -\frac{1}{2\pi^2} \int_0^1 dx x(1-x) \ln\left[\frac{m^2 - p^2x(1-x)}{m^2}\right]. \tag{16.10}$$

Since the maximum of the $x(1-x)$ is $\frac{1}{4}$, in the case if p^2 is space-like or if p^2 is time-like but $p < 2m$, the logarithmic contribution is always real, the contribution can only come from the region where $p \geq 2m$. Using the Eq. (24.19) of the book: $\ln(-A - i\varepsilon) = \ln A - i\pi$,

$$\mathrm{Im}[I(p^2 + i\varepsilon)] = -\frac{1}{2\pi} \int_0^1 dx x(1-x)\theta(p^2x(1-x) - m^2). \tag{16.11}$$

For a given p^2, the non-zero contribution of the step function comes from the region between $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\frac{m^2}{p^2}}$. Thus,

$$\begin{aligned}
\mathrm{Im}[I(p^2+i\varepsilon)] &= -\frac{1}{2\pi}\int_{\frac{1}{2}-\frac{1}{2}\sqrt{1-4\frac{m^2}{p^2}}}^{\frac{1}{2}+\frac{1}{2}\sqrt{1-4\frac{m^2}{p^2}}} dxx(1-x)\theta(p-2m) \\
&= -\frac{1}{2\pi}\int_{-\frac{1}{2}\sqrt{1-4\frac{m^2}{p^2}}}^{\frac{1}{2}\sqrt{1-4\frac{m^2}{p^2}}} du(\frac{1}{4}-u^2)\theta(p-2m) \\
&= -\frac{1}{8\pi}\sqrt{1-4\frac{m^2}{p^2}}\left[1-\frac{1}{3}(1-4\frac{m^2}{p^2})\right]\theta(p-2m) \\
&= -\frac{1}{12\pi}\sqrt{1-4\frac{m^2}{p^2}}\left[1+\frac{2m^2}{p^2}\right]\theta(p-2m),
\end{aligned} \tag{16.12}$$

where we have changed variable $u = x - \frac{1}{2}$ on the second line. Plugging this back to the Eq. (16.8), and also do a rotation $p \to -ip$ to get the branch cut, we shall arrive at

$$\begin{aligned}
V(r) &= -\frac{e^2}{4\pi r} - \frac{e^4}{2\pi^2 r}\int_{2m}^{\infty} dp\frac{1}{12\pi}\sqrt{1-4\frac{m^2}{p^2}}\left[1+\frac{2m^2}{p^2}\right]\frac{e^{-pr}}{p} \\
&= -\frac{e^2}{4\pi r}\left(1+\frac{e^2}{6\pi^2}\int_1^{\infty} dxe^{-2mrx}\frac{2x^2+1}{2x^4}\sqrt{x^2-1}\right),
\end{aligned} \tag{16.13}$$

where we changed the variable on the second line $p = 2mx$.

16.3

(a)

(b)

(c)

Chapter 17

The anomalous magnetic moment ✓

17.1

(a) The muon $g-2$ only receives correction from vertex correction. Given the Lagrangian of Eq. (17.33) of the book, there is only one vertex correction diagram involving the smuons and the photino that can contribute. It is shown in Fig. 17.1.

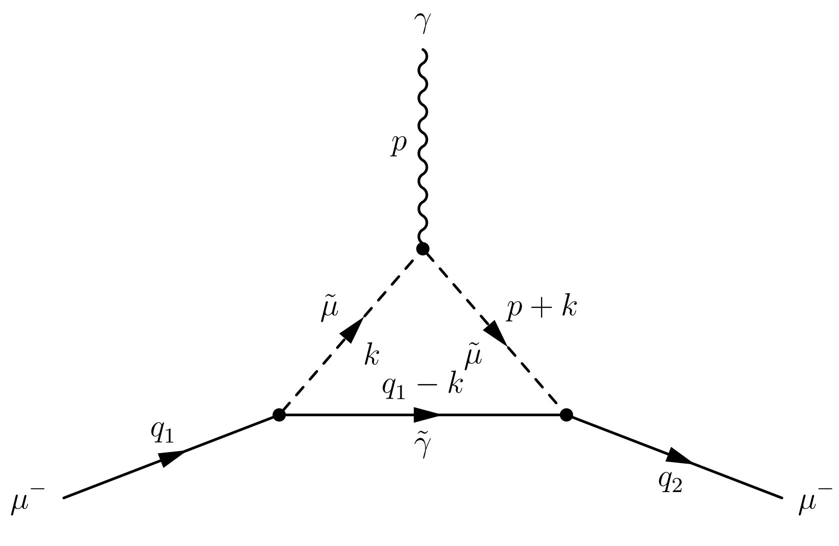


Fig. 17.1: Vertex correction of muon $g-2$ from SUSY particles.

The smuon is a scalar whose interaction with photon has exactly the same form and sign as the case of scalar QED Lagrangian described by the Eq. (9.11) of the book. Thus, we can use the results of Chapter 9 to determine its Feynman rule. Thus, we can write down the loop integral:

$$
\begin{aligned}
i\mathcal{M}_2^\mu &= -(ig)^3 \int \frac{d^4k}{(2\pi)^4} \bar{u}(q_2) \frac{i(\not{q}_1 - \not{k} + m_{\tilde{A}})}{(q_1-k)^2 - m_{\tilde{A}}^2 + i\varepsilon} \frac{i}{(p+k)^2 - m_{\tilde{\mu}}^2 + i\varepsilon} (p^\mu + 2k^\mu) \frac{i}{k^2 - m_{\tilde{\mu}}^2 + i\varepsilon} u(q_1) \\
&= g^3 \int \frac{d^4k}{(2\pi)^4} \bar{u}(q_2) \frac{(\not{q}_1 - \not{k} + m_{\tilde{A}})(p^\mu + 2k^\mu)}{[(q_1-k)^2 - m_{\tilde{A}}^2 + i\varepsilon][(p+k)^2 - m_{\tilde{\mu}}^2 + i\varepsilon][k^2 - m_{\tilde{\mu}}^2 + i\varepsilon]} u(q_1).
\end{aligned}
\tag{17.1}
$$

The minus sign in front of the first line comes from the interaction term of scalar QED. Using the Feynman parameters, the new denominator is the cube of

$$(k^\mu + yp^\mu - zq_1^\mu)^2 - \Delta + i\varepsilon \tag{17.2}$$

with

$$\Delta = -xyp^2 + zm_{\tilde{A}}^2 + (1-z)m_{\tilde{\mu}}^2 - z(1-z)m_\mu^2. \tag{17.3}$$

Shifting $k^\mu \to k^\mu - yp^\mu - zq_1^\mu$, the numerator becomes

$$\begin{aligned}
N^\mu &= \bar{u}(q_2)(\not{q}_1 - \not{k} + y\not{p} - z\not{q}_1 + m_{\tilde{A}})(p^\mu + 2k^\mu - 2yp^\mu + 2zq_1^\mu)u(q_1) \\
&= \bar{u}(q_2)[(1-z)m_\mu + \not{k} + m_{\tilde{A}}][(1-2y)p^\mu + 2k^\mu + 2zq_1^\mu]u(q_1) \\
&= \frac{k^2}{2}\bar{u}(q_2)\gamma^\mu u(q_1) + \bar{u}(q_2)[(1-z)m_\mu + m_{\tilde{A}}][(x-y)(q_2^\mu - q_1^\mu) + z(q_1^\mu + q_2^\mu)]u(q_1) \\
&= \left[\frac{k^2}{2} + 2z(1-z)m_\mu^2 + 2zm_\mu m_{\tilde{A}}\right]\bar{u}(q_2)\gamma^\mu u(q_1) \\
&\quad - i\left[z(1-z)m_\mu + zm_{\tilde{A}}\right]p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1) \\
&\quad + \left[(1-z)m_\mu + m_{\tilde{A}}\right](x-y)p^\mu \bar{u}(q_2)u(q_1).
\end{aligned} \tag{17.4}$$

Again, the p^μ term's integrand is antisymmetric under $x \leftrightarrow y$, but the integral measure is symmetric, so this term vanishes. The terms involving γ^μ only renormalize the electric charge. Therefore, for the magnetic moment, we shall have (replacing $g \to e$)

$$i\mathcal{M}_2^\mu = p_\nu \bar{u}(q_2)\sigma^{\mu\nu}u(q_1)\left[-2ie^3\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^4k}{(2\pi)^4}\frac{z(1-z)m_\mu + zm_{\tilde{A}}}{(k^2 - \Delta + i\varepsilon)^3}\right] + \cdots. \tag{17.5}$$

Then,

$$F_2(p^2) = \frac{2m_\mu}{e}(-2ie^3)\int_0^1 dxdydz\delta(x+y+z-1)\frac{-i[z(1-z)m_\mu + zm_{\tilde{A}}]}{32\pi^2[-xyp^2 + zm_{\tilde{A}}^2 + (1-z)m_{\tilde{\mu}}^2 - z(1-z)m_\mu^2]}. \tag{17.6}$$

At $p^2 = 0$,

$$\begin{aligned}
F_2(0) &= -\frac{m_\mu e^2}{8\pi^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{z(1-z)m_\mu + zm_{\tilde{A}}}{zm_{\tilde{A}}^2 + (1-z)m_{\tilde{\mu}}^2 - z(1-z)m_\mu^2} \\
&= -\frac{\alpha}{2\pi}m_\mu\int_0^1 dz\frac{z(1-z)^2 m_\mu + z(1-z)m_{\tilde{A}}}{zm_{\tilde{A}}^2 + (1-z)m_{\tilde{\mu}}^2 - z(1-z)m_\mu^2}.
\end{aligned} \tag{17.7}$$

Assuming $m_\mu \ll m_{\tilde{\mu}} \approx m_{\tilde{A}}$, this becomes

$$\begin{aligned}
F_2(0) &= -\frac{\alpha}{2\pi}\frac{m_\mu}{m_{\tilde{\mu}}}\int_0^1 dzz(1-z) \\
&= -\frac{\alpha}{12\pi}\frac{m_\mu}{m_{\tilde{\mu}}}.
\end{aligned} \tag{17.8}$$

Then, since $g = 2 + 2F_2(0)$, the contribution to magnetic moment caused by smuon and photino is

$$g_{\text{SUSY}} = g_{\text{SM}} - \frac{\alpha}{6\pi}\frac{m_\mu}{m_{\tilde{\mu}}}. \tag{17.9}$$

Observe that there is an interesting limit if the supersymmetry is restored[1] on Eq. (17.7) such that the mass of each particles is the same as their super-partner's ($m_{\tilde{\mu}} = m_\mu$ and $m_{\tilde{A}} = m_A = 0$), we have

$$\begin{aligned} F_2(0) &= -\frac{\alpha}{2\pi}\int_0^1 dz \frac{z(1-z)^2 m_\mu^2}{(1-z)^2 m_\mu^2} \\ &= -\frac{\alpha}{2\pi}\int_0^1 dz z \\ &= -\frac{\alpha}{4\pi}. \end{aligned} \tag{17.10}$$

Compared this with the Eq. (17.31) of the book, we can observe that this is exactly half of the contribution of the SM muon-photon loop. In fact, there should be two smuons $\tilde{\mu}_1$ and $\tilde{\mu}_2$ since in SUSY, the Fermionic degree of freedom is equal to the Bosonic degree of freedom within a supermultiplet. Therefore, each fermion actually has two scalar super-partners corresponding to the left-handed and right-handed chirality. As result, one should add a factor of 2 to the above result, and concludes that **in the limit of unbroken SUSY**:

$$g = 2 + 2F_2^{\text{SM}}(0) + 2F_2^{\text{SUSY}}(0) = 2 + 2\times\frac{\alpha}{2\pi} - 2\times 2\times\frac{\alpha}{4\pi} = 2. \tag{17.11}$$

The profound result is that in the SUSY limit, the loop corrections from the SM particles cancel exactly with the loop corrections from their super-partners. The opposite contribution is really on a fundamental ground due to spin statistics and is the reason why the SUSY can stabilize the mass of Higgs boson from receiving large radiative correction. Please also see Ref. [5] for a more general derivation on how SUSY precludes the Pauli term as such term is not SUSY-invariant. Also note this is, on a even deeper level due to SUSY non-renormalization theorem.

(b) I found the parts (*b*) and (*c*) of this problem are not sensible. The reason is that $a_\mu^{\text{EXP}} > a_\mu^{\text{SM}}$ where $a_\mu \equiv \frac{g_\mu - 2}{2}$. Yet, from my discussion above, the SUSY contribution from smuon-photino should further diminish the SM muon-photon contribution, which is an essential point. Notice this does not mean the SUSY has already been ruled out since there are other SUSY contributions to the muon anomalous magnetic moment that increases the anomaly. The picture in Minimal Supersymmetric Standard Model (MSSM) is actually a bit more complicated. After electroweak symmetry breaking, the particles having the same quantum number mix together. Therefore, the photino is not a physical states. Instead, the neutral superpartners of SM gauge bosons (electroweakino) mix and form four neutralinos (and usually, one of them has a "negative" mass, which could also flip the sign of the contribution of the Eq. (17.7) if this neutralino is much lighter than the others). In MSSM, the muon also has trilinear interaction with a sneutrino and a chargino and thus, there are

[1] SUSY must be broken for the particles and their super-partners to have different masses.

also contributions from the chargino-sneutrino-chargino (which is usually opposite to the above smuon-neutralino-smuon contribution).

It might also be interesting to notice that while $a_\mu^{\mathrm{EXP}} > a_\mu^{\mathrm{SM}}$[2], the electron $a_e^{\mathrm{EXP}} < a_e^{\mathrm{SM}}$[7].

(c) See discussions in (b).

[2] At the point of writing, this anomaly is already at a level of 5.1σ [6]. The anomaly, however is debatable as the anomaly seems to be consistent with the lattice results.

Chapter 18

Mass renormalization ✓

18.1

(a) At one-loop level of scalar QED, a scalar has two kinds of self-energy graphs shown in Fig. 18.1.

Let m denotes the mass of the scalar. The graph (a) can be evaluated as

$$\begin{aligned} i\Sigma_{2,a}(p^2) &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4}(p^\mu + k^\mu)\frac{i}{k^2 - m^2 + i\varepsilon}(p^\mu + k^\mu)\frac{-i}{(k-p)^2 + i\varepsilon} \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{p^2 + k^2 + 2p\cdot k}{[(k^2 - m^2)(1-x) + (p-k)^2 x + i\varepsilon]^2}. \end{aligned} \tag{18.1}$$

Shifting $k \to k + px$ gives

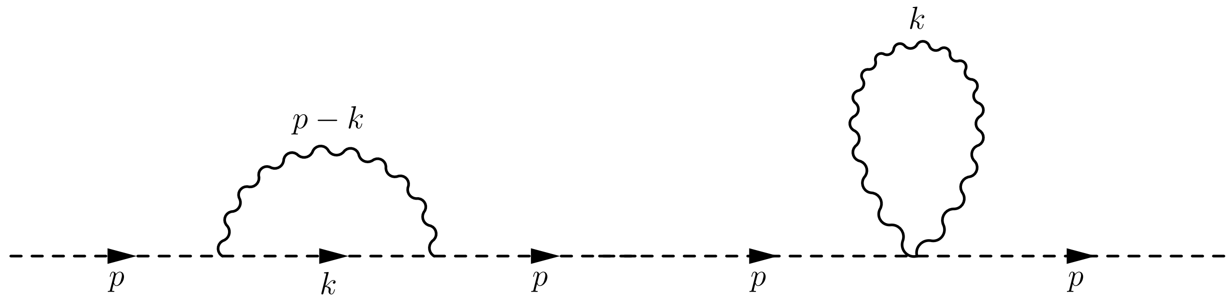
$$i\Sigma_{2,a}(p^2) = -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{k^2 + p^2(1+x)^2}{[k^2 - \Delta + i\varepsilon]^2}, \tag{18.2}$$

where $\Delta = (1-x)(m^2 - p^2x)$ and we have dropped the terms linear in k in the numerator since it is odd under $k \to -k$ and their integral therefore vanish.

In dimensional regularization, in $d = 4 - \varepsilon$ dimensions, the loop is

$$\begin{aligned} \Sigma_{2,a}(p^2) &= ie^2\mu^{4-d}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{k^2 + p^2(1+x)^2}{(k^2 - \Delta + i\varepsilon)^2} \\ &= -\frac{e^2}{(4\pi)^{d/2}}\mu^{4-d}\int_0^1 dx \left[-\frac{d}{2}\frac{1}{\Delta^{1-\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right) + p^2(1+x)^2\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\right] \\ &= -\frac{e^2}{(4\pi)^{d/2}}\mu^{4-d}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx \left[-\frac{d}{2}\frac{\Delta}{\Delta^{2-\frac{d}{2}}}\frac{1}{1-\frac{d}{2}} + p^2(1+x)^2\frac{1}{\Delta^{2-\frac{d}{2}}}\right] \\ &= -\frac{e^2}{(4\pi)^{d/2}}\mu^{4-d}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx \frac{\frac{d}{d-2}(1-x)(m^2 - p^2x) + p^2(1+x)^2}{\left[(1-x)(m^2-p^2x)\right]^{2-\frac{d}{2}}}, \end{aligned} \tag{18.3}$$

where we used $\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$. Expanding $d = 4-\varepsilon$ we get, in the $\varepsilon \to 0$ limit, first notice that $\frac{d}{d-2} = 1 + \frac{2}{d-2} = 1 + \frac{1}{1-\frac{\varepsilon}{2}} \to 2 + \frac{\varepsilon}{2} + \mathcal{O}((\frac{\varepsilon}{2})^2)$ and thus the whole



(a) (b)

Fig. 18.1: Scalar QED self-energy graphs for a scalar.

integral becomes

$$\begin{aligned}\Sigma_{2,a}(p^2) &= -\frac{\alpha}{4\pi}\int_0^1 dx\left[(2+\frac{\varepsilon}{2})(1-x)(m^2-p^2x)+p^2(1+x)^2\right]\left[\frac{2}{\varepsilon}+\ln\frac{\tilde{\mu}^2}{(1-x)(m^2-p^2x)}\right]\\ &= -\frac{\alpha}{4\pi}\left\{\frac{2m^2}{\varepsilon}+\frac{4p^2}{\varepsilon}+\frac{m^2}{2}-\frac{p^2}{6}\right.\\ &\qquad\left.+\int_0^1 dx[2(1-x)m^2+(3x^2+1)p^2]\ln\frac{\tilde{\mu}^2}{(1-x)(m^2-p^2x)}\right\}.\end{aligned} \tag{18.4}$$

The graph (b) can be evaluated (in d dimensions) as

$$i\Sigma_{2,b}(p^2) = 2ie^2 d\int\frac{d^dk}{(2\pi)^d}\frac{-i}{k^2+i\varepsilon} = 0, \tag{18.5}$$

where we used the contraction of the metric tensor $g^{\mu\nu}g_{\mu\nu} = d$. Now, this is a scaleless integral and is both UV and IR divergent and thus formally **vanishes in dimensional regularization**. One can refer to the discussion of Section 26.4.3 of the book. The graph therefore has no contribution to the radiative correction. From now on, we will drop the subscript a from $\Sigma_{2,a}(p^2)$ since graph (a) is the only self-energy graph that can contribute under dimensional regularization.

(b) The bare Green's function the renormalized Green's function are still related as

$$iG^R(p^2) = \frac{1}{1+\delta_2}iG^{\text{bare}}(p^2) = \frac{i}{p^2-m_R^2+\delta_2p^2-(\delta_2+\delta_m)m_R^2+\Sigma_2(p^2)+\cdots}, \tag{18.6}$$

where the mass counterterm is defined through $m_0^2 = Z_m m_R^2$. The pole mass is defined by the pole of the Green's function:

$$\Sigma_R(m_P^2) = m_R^2 - m_P^2, \tag{18.7}$$

where $\Sigma_R(p^2) = \Sigma_2(p^2) + \delta_2 p^2 - (\delta_m+\delta_2)m_R^2 + \mathcal{O}(e^4)$.

(c) • **On-shell:**

For the on-shell subtraction scheme, $m_R^2 = m_P^2$ and thus,

$$\begin{aligned}\delta_m &= \frac{1}{m_P^2}\Sigma_2(m_P^2)\\ &= -\frac{3\alpha}{4\pi}\left(\frac{2}{\varepsilon}+\ln\frac{\tilde{\mu}^2}{m_P^2}+\frac{7}{3}\right).\end{aligned} \tag{18.8}$$

It should be noticed that the $\Sigma_2'(m_P)$, just like the Eq. (18.50) of the book, has infrared divergence, and should be regulated with a photon mass m_γ. This just changes Δ to $\Delta = (1-x)(m_P^2 - p^2x) + xm_\gamma^2$ (since we will only keep the leading terms in m_γ, which is from logarithmic term and the m_γ from outside the logarithmic term must be in higher orders) so that

$$\begin{aligned}\Sigma_2(p^2) = -\frac{\alpha}{4\pi}\Bigg\{&\frac{2m^2}{\varepsilon}+\frac{4p^2}{\varepsilon}+\frac{m^2}{2}-\frac{p^2}{6}\\ &+\int_0^1 dx[2(1-x)m^2+(3x^2+1)p^2]\ln\frac{\tilde{\mu}^2}{(1-x)(m^2-p^2x)+xm_\gamma^2}\Bigg\}.\end{aligned} \tag{18.9}$$

Thus,

$$\begin{aligned}\delta_2 &= -\Sigma_2'(m_P^2)\\ &= \frac{\alpha}{4\pi}\left(\frac{4}{\varepsilon}-\frac{1}{6}+2\ln\frac{\tilde{\mu}^2}{m_P^2}+\frac{17}{3}-\frac{11}{2}-2\ln\frac{m_\gamma^2}{m_P^2}\right)\\ &= \frac{\alpha}{2\pi}\left(\frac{2}{\varepsilon}+\ln\frac{\tilde{\mu}^2}{m_\gamma^2}\right).\end{aligned} \tag{18.10}$$

• **$\overline{\text{MS}}$:**

For $\overline{\text{MS}}$, the counterterms are simply just the divergent parts plus the constant terms that convert $\tilde{\mu}$ back to μ. Therefore,

$$\delta_m = -\frac{3\alpha}{4\pi}\left(\frac{2}{\varepsilon}+\ln\left(4\pi e^{-\gamma_E}\right)\right), \tag{18.11}$$

$$\delta_2 = \frac{\alpha}{2\pi}\left(\frac{2}{\varepsilon}+\ln\left(4\pi e^{-\gamma_E}\right)\right). \tag{18.12}$$

Chapter 19

Renormalized perturbation theory

19.1

The bare Lagrangian of scalar QED is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^0_\nu - \partial_\nu A^0_\mu) - \phi^{0*}(\Box + m_0^2)\phi^0 - ie_0 A^0_\mu \left[\phi^{0*}\left(\partial_\mu \phi^0\right) - \left(\partial_\mu \phi^{0*}\right)\phi^0\right] + e_0^2 (A^0_\mu)^2 |\phi^0|^2. \quad (19.1)$$

Renormalizing the field strength, mass, and the charge ("R" omitted for the renormalized fields):

$$\phi^0 = \sqrt{Z_2}\phi,\ A^0_\mu = \sqrt{Z_3}A_\mu,\ m_0 = Z_m m_R,\ e_0 = Z_e e_R. \quad (19.2)$$

Also, just like the case in spinor QED, we can also define a $Z_1 \equiv Z_e Z_2 \sqrt{Z_3}$, which encodes the renormalization of the 3-point interaction. Notice that the renormalization of the 4-point interaction is completely fixed by orther renormalization factors:

$$Z_{\text{4-point}} = Z_e^2 Z_2 Z_3. \quad (19.3)$$

Expanding the renormalizations around their classical tree-level values let us to extract the counterterms. For the 4-point interaction, its counterterm is totally fixed by:

$$\delta_{\text{4-point}} = 2\delta_e + \delta_2 + \delta_3 + \mathcal{O}(e_R^4) = 2\delta_1 - \delta_2 + \mathcal{O}(e_R^4), \quad (19.4)$$

where we used the fact that $\delta_e = \delta_1 - \delta_2 - \frac{1}{2}\delta_3 + \mathcal{O}(e_R^4)$. Since $\delta_1 = \delta_2$ (which will be proven in the Problem 19.2 and explicitly calculate below),

$$\delta_{\text{4-point}} = \delta_1. \quad (19.5)$$

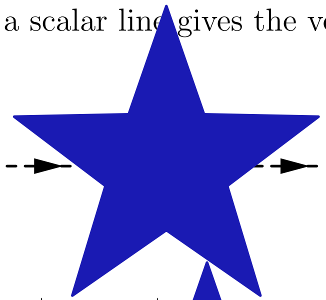
After the expansions, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 - \phi^*(\Box + m_R^2)\phi - ie_R A_\mu \left[\phi^*\left(\partial_\mu\phi\right) - \left(\partial_\mu\phi^*\right)\phi\right] + e_R^2 A_\mu^2 |\phi|^2 \\ &- \frac{1}{4}\delta_3 F_{\mu\nu}^2 - \delta_2 \phi^* \Box \phi - (\delta_m + \delta_2) m_R |\phi|^2 - ie_R \delta_1 A_\mu \left[\phi^*\left(\partial_\mu\phi\right) - \left(\partial_\mu\phi^*\right)\phi\right] + \delta_1 e_R^2 A_\mu^2 |\phi|^2.\end{aligned} \quad (19.6)$$

We can read off the Feynman rules from the counterterms.

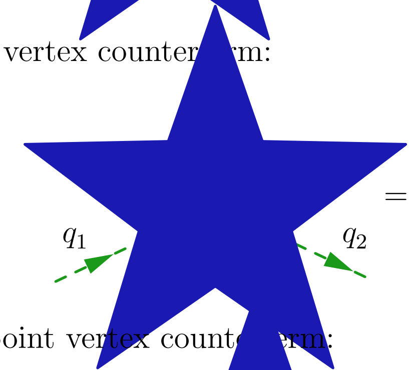
A counterterm on a photon line is the same as the Eq. (19.14) of the book.

A counter term on a scalar line gives the vertex



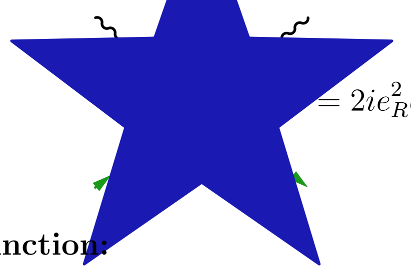
$$= i(p^2\delta_2 - (\delta_m + \delta_2)m_R^2). \tag{19.7}$$

There is a 3-point vertex counterterm:



$$= -ie_R\delta_1(q_1^\mu + q_2^\mu). \tag{19.8}$$

There is also a 4-point vertex counterterm:



$$= 2ie_R^2\delta_1 g_{\mu\nu}. \tag{19.9}$$

- **Photon 2-point function:**

  We will stick with the dimensional regularization for this question. Starting with renormalizing the 2-point functions. The photon self-energy graph in scalar QED has already been evaluated in the Chapter 16 of the book and Problem 16.1. We quote the results from the Eq. (16.39) of the book (with appropriate factor out of the tree-level propagator):

$$\Pi_2(p^2) = \frac{1}{8\pi^2}\int_0^1 dx x(2x-1)\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2x(1-x)}\right)\right]. \tag{19.10}$$

  At order e_R^2,

$$\Pi(p^2) = e_R^2\Pi_2(p^2) + \delta_3 + \cdots. \tag{19.11}$$

  The on-shell renormalization condition for the photon in scalar QED is then still

$$\Pi(0) = 0. \tag{19.12}$$

  Thus,

$$\delta_3 = -e_R^2\Pi_2(0) = -\frac{e_R^2}{8\pi^2}\left(\frac{1}{3} + \frac{1}{6}\ln\frac{\tilde{\mu}^2}{m_R^2}\right). \tag{19.13}$$

- **Scalar 2-point function:**

  Then, there is the scalar 2-point function, which we have already evaluated in the Problem 18.1 and we just quote results from Eq. (18.8) and Eq. (18.10) and remember the on-shell condition that $m_R^2 = m_P^2$:

$$\delta_m = -\frac{3e_R^2}{16\pi}\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{m_R^2} + \frac{7}{3}\right). \tag{19.14}$$

and

$$\delta_2 = \frac{e_R^2}{8\pi}\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{m_\gamma^2}\right). \tag{19.15}$$

- **3-point function:**

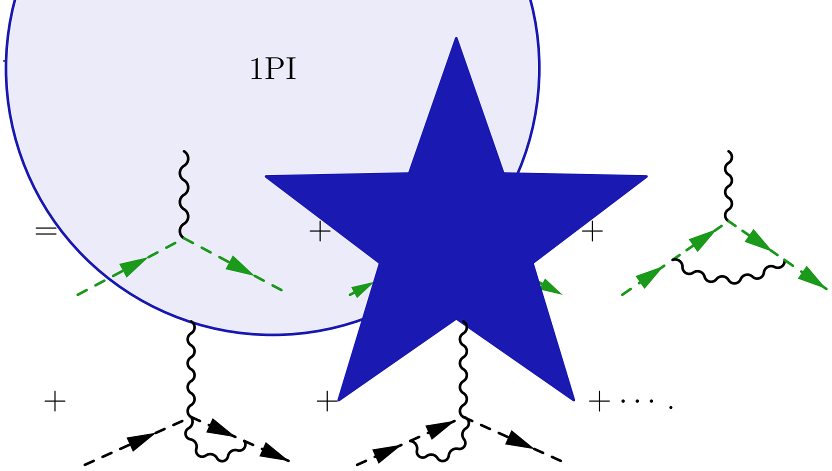
  Next, we shall renormalize the three-point functions of scalar QED. Notice that the in scalar QED, there is no Pauli moment term, the form factor is related with the vertex correction as

  $$\Gamma^\mu(p) = F_1(p^2)(q_1^\mu + q_2^\mu) \tag{19.16}$$

  with $p^\mu = q_2^\mu - q_1^\mu$ and $F_1(p^2) = 1 + f(p^2) + \delta_1 + \mathcal{O}(e_R^4)$. At leading order:

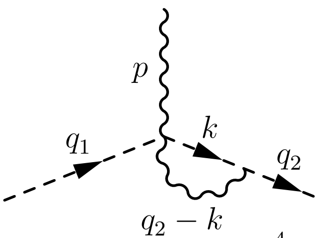
  $$F_1(p^2) = 1. \tag{19.17}$$

  At NLO e_R^2, th... contributions from the following amputated graphs:



(19.18)

Evaluating



$$
\begin{aligned}
i\mathcal{M}^\mu &= \\
&= (-ie)(2ie^2 g^{\mu\nu}) \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{(q_2-k)^2 + i\varepsilon}(q_2^\alpha + k^\alpha)\frac{i}{k^2 - m^2 + i\varepsilon} \\
&= 2e^3 \int \frac{d^4k}{(2\pi)^4} \frac{q_2^\mu + k^\mu}{[(q_2-k)^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \\
&= 2e^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{q_2^\mu + k^\mu}{[(k^2 - m^2)(1-x) + (q_2-k)^2 x + i\varepsilon]^2}
\end{aligned} \tag{19.19}
$$

Shifting $k^\mu \to k^\mu + q_2^\mu x$ and dropping terms linear in k^μ we get

$$\mathcal{M}^\mu = -2ie^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{(1+x)q_2^\mu}{[k^2 - \Delta + i\varepsilon]^2}, \tag{19.20}$$

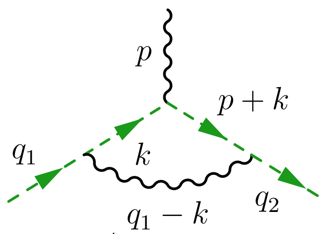
where we use the on-shellness of the scalar and $\Delta = m^2(1-x)^2$. Similarly, the graph that has the internal photon line emitting from the incoming scalar and ending at the four-point vertex should contribute as

$$\mathcal{M}^\mu = -2ie^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{(1+x)q_1^\mu}{[k^2-\Delta+i\varepsilon]^2}, \tag{19.21}$$

Reading off the coefficients of $(q_1^\mu + q_2^\mu)$, we observe that together, these two graphs contribute to $f(p^2)$ as

$$\begin{aligned} f(p)^2 &= 2ie_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1+x}{[k^2-\Delta+i\varepsilon]^2} \\ &= -2\frac{e_R^2}{(4\pi)^{d/2}}\mu^{4-d} \int_0^1 dx(1+x)\left(\frac{1}{\Delta^{2-\frac{d}{2}}}\right)\Gamma\left(\frac{4-d}{2}\right) \\ &= -\frac{e_R^2}{8\pi^2}\left(\frac{3}{\varepsilon}+\frac{3}{2}\ln\frac{\tilde{\mu}^2}{m_R^2}+\frac{7}{2}\right). \end{aligned} \tag{19.22}$$

Then, there is also the

$i\mathcal{M}^\mu =$ 

$$\begin{aligned} &= (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\alpha}}{(q_1-k)^2+i\varepsilon}(q_2^\nu + p^\nu + k^\nu)\frac{i}{(p+k)^2-m^2+i\varepsilon}(p^\mu+2k^\mu)\frac{i}{k^2-m^2+i\varepsilon}(k^\alpha+q_1^\alpha) \\ &= -2e^3 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{(p^\mu+2k^\mu)(k^2+2k\cdot q_2+2q_2\cdot q_1-m^2)}{[(k+yp-zq_1)^2-\Delta+i\varepsilon]^3}, \end{aligned} \tag{19.23}$$

where $\Delta = -xyp^2 + (1-z)^2m^2$. Shifting $k^\mu \to k^\mu - yp^\mu + zq_1^\mu$, and also remember that $p\cdot q_1 = q_1\cdot q_2 - m^2 = -p\cdot q_2 = -\frac{p^2}{2}$, the numerator becomes

$$\begin{aligned} N^\mu &= [p^\mu(1-2y)+2zq_1^\mu+2k^\mu] \\ &\quad\times \left[k^2 + k^\nu(-2yp^\nu+2zq_1^\nu+p^\nu+q_1^\nu+q_2^\nu)+p^2(y^2-y-1+yz-z)+m^2(z^2+z+1+z)\right] \\ &= [z(q_1^\mu+q_2^\mu)+(x-y)p^\mu+2k^\mu] \\ &\quad\times\left[k^2+k^\nu((z+1)(q_1^\nu+q_2^\nu)+(x-y)p^\nu)-p^2(1+(1-y)(1-x))+m^2(z+1)^2\right] \\ &= 2k^\mu k^\nu\left[(z+1)(q_1^\nu+q_2^\nu)+(x-y)p^\nu\right]+k^2\left[z(q_1^\mu+q_2^\mu)+(x-y)p^\mu\right] \\ &\quad+\left[z(q_1^\mu+q_2^\mu)+(x-y)p^\mu\right]\left[-p^2(1+(1-y)(1-x))+m^2(z+1)^2\right]+\cdots \\ &= k^2\left[\left(\frac{2}{d}+z\frac{2}{d}+z\right)(q_1^\mu+q_2^\mu)+\left(\frac{2}{d}+z\right)(x-y)p^\mu\right] \\ &\quad+\left[z(q_1^\mu+q_2^\mu)+(x-y)p^\mu)\right]\left[m^2(1+z)^2-p^2(1+(1-x)(1-y))\right]+\cdots, \end{aligned} \tag{19.24}$$

where $\cdots$ contain the terms that have odd number of factors of k^μ which shall vanish after integration. Also notice that the $(x-y)p^\mu$ is antisymmetric with respect to $x \leftrightarrow y$, but the rest of the integral is symmetric so any terms involved with it also vanishes after integration. The only non-vanishing parts of the numerators are then

$$N^\mu = (q_1^\mu + q_2^\mu)\left[k^2\left(\frac{2}{d} + z\frac{2}{d} + z\right) + z(m^2(1+z)^2 - p^2(1+(1-x)(1-y)))\right]. \quad (19.25)$$

Reading off the coefficients and taking the limit $p^2 \to 0$, we see its contribution to $f(p^2)$ is

$$f(0) = -2ie_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdydz\delta(x+y+z-1)\frac{k^2\left(\frac{2}{d} + z\frac{2}{d} + z\right) + m_R^2 z(1+z)^2}{\left[k^2 - (1-z)^2 m_R^2 + i\varepsilon\right]^3}. \quad (19.26)$$

The k^2 term is UV-divergent, and we can use the dimensional regularization and then expand in $d = 4 - \varepsilon$ to regulate that. This part is

$$\begin{aligned}
&-2ie_R^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dxdydz\delta(x+y+z-1)\frac{k^2\left(\frac{2}{d} + z\frac{2}{d} + z\right)}{\left[k^2 - (1-z)^2 m_R^2 + i\varepsilon\right]^3} \\
&= -2ie_R^2\mu^{4-d}\frac{i}{(4\pi)^{d/2}}\frac{d}{4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2+2z+zd}{d}((1-z)^2 m_R^2)^{\frac{d}{2}-2}\Gamma\left(\frac{4-d}{2}\right) \\
&= \frac{e_R^2}{8\pi^2}\int_0^1 dxdydz\delta(x+y+z-1)\left[\frac{1+3z}{\varepsilon} + \frac{1+3z}{2}\ln\frac{\tilde{\mu}^2}{(1-z)^2 m_R^2} - \frac{z}{2}\right] \\
&= \frac{e_R^2}{8\pi^2}\int_0^1 dz\left[\frac{(1+3z)(1-z)}{\varepsilon} + \frac{(1+3z)(1-z)}{2}\ln\frac{\tilde{\mu}^2}{(1-z)^2 m_R^2} - \frac{(1-z)z}{2}\right] \\
&= \frac{e_R^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \frac{1}{2}\ln\frac{\tilde{\mu}^2}{m_R^2} + \frac{7}{12}\right).
\end{aligned} \quad (19.27)$$

The rest terms are UV finite but IR divergent, so we can set $d = 4$ in them and add a photon mass which changes Δ to $\Delta = (1-z)^2 m_R^2 + z m_\gamma^2$ when $p^2 = 0$. Then, we evaluate

$$\begin{aligned}
&-2ie_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdydz\delta(x+y+z-1)\frac{m_R^2 z(1+z)^2}{\left[k^2 - (1-z)^2 m_R^2 - z m_\gamma^2 + i\varepsilon\right]^3} \\
&= -\frac{e_R^2}{16\pi^2}\int_0^1 dxdydz\delta(x+y+z-1)\frac{m_R^2 z(1+z)^2}{(1-z)^2 m_R^2 + z m_\gamma^2} \\
&= \frac{e_R^2}{8\pi^2}\left(\ln\frac{m_\gamma^2}{m_R^2} + \frac{35}{12}\right)
\end{aligned} \quad (19.28)$$

Summing over Eq. (19.22), Eq. (19.27), and Eq. (19.28), we get

$$\begin{aligned}
f(0) &= \frac{e_R^2}{8\pi^2}\left(-\frac{2}{\varepsilon} - \ln\frac{\tilde{\mu}^2}{m_R^2} + \ln\frac{m_\gamma^2}{m_R^2} - \frac{7}{2} + \frac{7}{12} + \frac{35}{12}\right) \\
&= \frac{e_R^2}{8\pi^2}\left(-\frac{2}{\varepsilon} - \ln\frac{\tilde{\mu}^2}{m_\gamma^2}\right).
\end{aligned} \quad (19.29)$$

Thus,

$$\delta_1 = -f(0) = \frac{e_R^2}{8\pi^2}\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{m_\gamma^2}\right). \tag{19.30}$$

Compared this with the Eq. (19.15), we observe that $\delta_1 = \delta_2$ is also true in scalar QED. We shall prove $Z_1 = Z_2$ in Problem 19.2.

19.2

Chapter 20

Infrared divergences

20.1

$$\begin{aligned}\int d\Pi_{\text{LIPS}} &= \prod_{j=3,4,\gamma} \int \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_{p_j}} (2\pi)^4 \delta^4(p - p_3 - p_4 - p_\gamma) \\ &= \left(\frac{1}{2\pi}\right)^5 \int \frac{d^3 p_3}{2E_3} \int \frac{d^3 p_4}{2E_4} \frac{1}{2E_\gamma} \delta^4(Q - E_3 - E_4 - E_\gamma),\end{aligned} \tag{20.1}$$

where we integrate over the 3-momenta $\vec{p}_\gamma$. Let $x_1 = 2\frac{E_4}{Q}$, $x_2 = 2\frac{E_3}{Q}$, $x_\gamma = 2\frac{E_\gamma}{Q} - \beta$, and $x'_\gamma = 2\frac{E_\gamma}{Q} = x_\gamma + \beta$. Then, $d^3 p_i = d\Omega_{p_i} p_i^2 dp_i = d\Omega \frac{Q^3}{8} x_i^2 dx_i$. This leads to

$$\begin{aligned}\int d\Pi_{\text{LIPS}} &= \left(\frac{1}{2\pi}\right)^5 \frac{Q^3}{64} \int d\Omega_{p_4} x_1 dx_1 \int d\Omega_{p_3} x_2 dx_2 \frac{1}{x'_\gamma} \delta^4(Q - E_3 - E_4 - E_\gamma) \\ &= \left(\frac{1}{2\pi}\right)^5 \frac{Q^2}{32} \int d\Omega_{p_4} x_1 dx_1 \int d\Omega_{p_3} x_2 dx_2 \frac{1}{x'_\gamma} \delta^4(2 - x_2 - x_1 - x'_\gamma).\end{aligned} \tag{20.2}$$

The x'_γ has is an implicit function of the 3-momenta of $\vec{p}_3$ and $\vec{p}_4$:

$$x'_\gamma = \frac{2E_\gamma}{Q} = \frac{2}{Q}\sqrt{(\vec{p}_3 + \vec{p}_4)^2 + m_\gamma^2} = \sqrt{x_1^2 + x_2^2 - 2x_1 x_2 \cos\theta + 4\beta}, \tag{20.3}$$

where θ is the angle between the $\vec{p}_3$ and $\vec{p}_4$. Then,

$$\frac{dx'_\gamma}{d\cos\theta} = -\frac{x_1 x_2}{x'_\gamma}. \tag{20.4}$$

Also, notice that the boundary of x'_γ is set by $\theta = 0$ and π, but the delta function also forces that $x'_\gamma = 2 - x_1 - x_2$. For $\theta = 0$, we thus have

$$\begin{aligned}(x'_\gamma)^2 &= (x_1 - x_2)^2 + 4\beta \\ (2 - (x_1 + x_2))^2 &= (x_1 - x_2)^2 + 4\beta \\ 1 + x_1 x_2 - x_1 - x_2 &= \beta \\ (1 - x_1)(1 - x_2) &= \beta.\end{aligned} \tag{20.5}$$

For $\theta = \pi$, we have

$$\begin{aligned}
(x'_\gamma)^2 &= (x_1 + x_2)^2 + 4\beta \\
(2 - (x_1 + x_2))^2 &= (x_1 + x_2)^2 + 4\beta \\
1 - x_1 - x_2 &= \beta \\
x_1 + x_2 &= 1 - \beta.
\end{aligned} \tag{20.6}$$

$$\begin{aligned}
\int d\Pi_{\text{LIPS}} &= \frac{Q^2}{128\pi^3} \int x_1 dx_1 \int x_2 dx_2 \int d(\cos\theta) \frac{1}{x'_\gamma} \delta^4(2 - x_2 - x_1 - x'_\gamma) \\
&= \frac{Q^2}{128\pi^3} \int dx_1 dx_2 dx'_\gamma \delta^4(x_1 + x_2 + x'_\gamma - 2) \\
&= \frac{Q^2}{128\pi^3} \int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\frac{\beta}{1-x_1}} dx_2.
\end{aligned} \tag{20.7}$$

Here, we arrived at the phase space formula in Eq. (20.42) of the book.

20.2

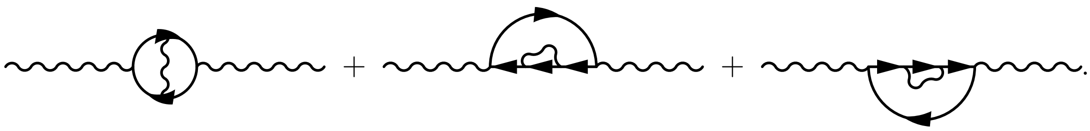
Chapter 21

Renormalizability

21.1

The 1PI diagrams for the superficially divergent amplitudes are

- $\langle AA \rangle$:

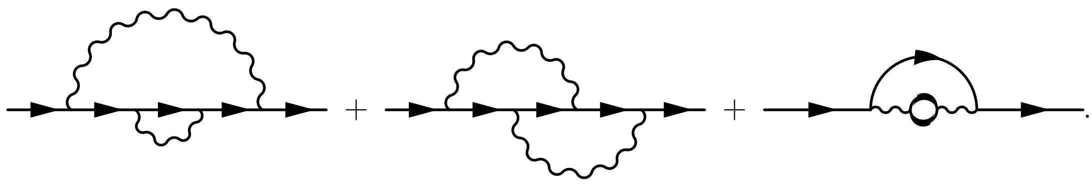
$\langle AA \rangle \; =$ 

(21.1)

All these 1PI diagrams go like

$$\langle AA \rangle \sim \int \frac{d^8 k}{(2\pi)^8} \left(\frac{1}{\not{k}}\right)^4 \left(\frac{1}{k^2}\right) \sim \Lambda^2 \implies D = 2. \tag{21.2}$$

- $\langle \bar{\psi}\psi \rangle$:

$\langle \bar{\psi}\psi \rangle \; =$ 

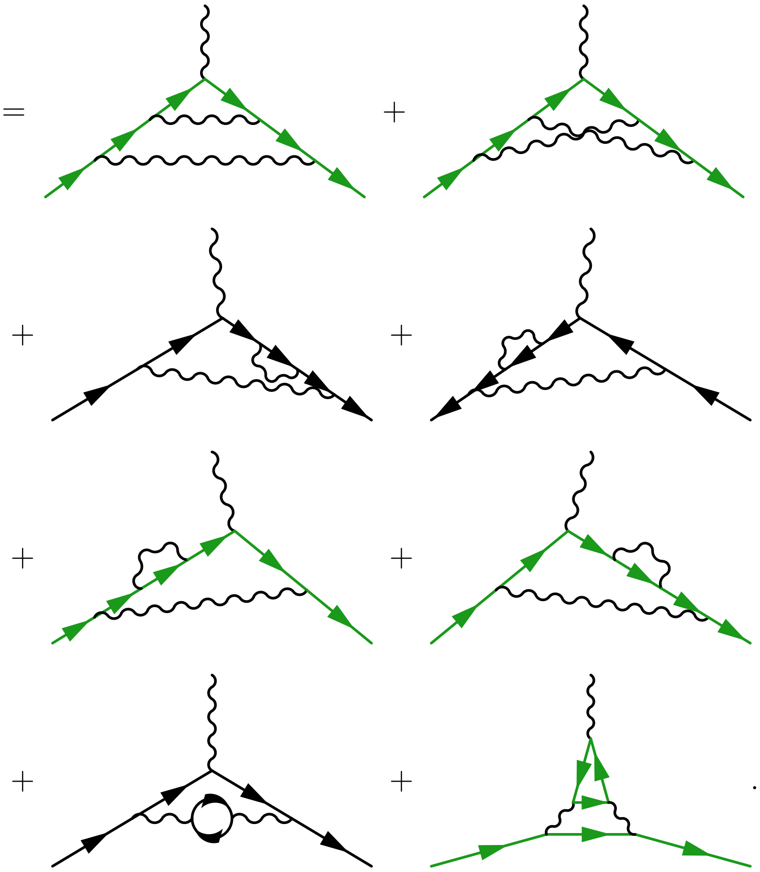
(21.3)

All these 1PI diagrams go like

$$\langle \bar{\psi}\psi \rangle \sim \int \frac{d^8 k}{(2\pi)^8} \left(\frac{1}{\not{k}}\right)^3 \left(\frac{1}{k^2}\right)^2 \sim \Lambda^1 \implies D = 1. \tag{21.4}$$

- $\langle\bar{\psi}\psi A\rangle$:

$\langle\bar{\psi}\psi A\rangle =$



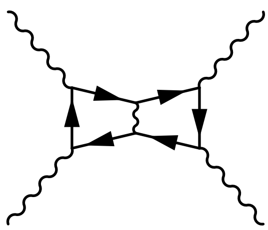
(21.5)

All these 1PI diagrams go like (the last diagram actually has an ABJ anomaly due to the triangle sub-diagram and thus actually vanishes)

$$\langle\bar{\psi}\psi A\rangle \sim \int \frac{d^8k}{(2\pi)^8}\left(\frac{1}{\not{k}}\right)^4\left(\frac{1}{k^2}\right)^2 \sim \Lambda^0 \implies D = 0. \tag{21.6}$$

- $\langle AAAA\rangle$:

$\langle AAAA\rangle =$



$+ \cdots,$ (21.7)

where the $\cdots$ contain all the other variations. They are all characterized by attaching a photon line onto and closed on a fermion line or onto and connects to another fermion line. Also, there can be a permutation of the interchanging the final states. All these 1PI diagrams go like

$$\langle AAAA\rangle \sim \int \frac{d^8k}{(2\pi)^8}\left(\frac{1}{\not{k}}\right)^6\left(\frac{1}{k^2}\right) \sim \Lambda^0 \implies D = 0. \tag{21.8}$$

These have exhausted the superficially divergent amplitudes in QED at 2-loops. Notice from the explicit enumeration, the superficial degree of divergence does not change from that of their corresponding 1-loop result, as can be compared with the Table 21.1 of the book. Thus, the same four counterterms must be able to remove all of the UV divergences. For higher point functions, there are no superficially divergent amplitudes and the argument shall just be the same as the case in 1-loop.

21.2

Chapter 22

Non-renormalizable theories

22.1

The first order perturbation correction in quantum mechanics due to this term is:

$$
\begin{aligned}
\left\langle \psi^{(0)} \middle| \frac{\vec{p}^6}{16m^5} \middle| \psi^{(0)} \right\rangle &= \frac{1}{2m^2} \left\langle \psi^{(0)} \middle| \left(\frac{\vec{p}^2}{2m}\right)^3 \middle| \psi^{(0)} \right\rangle \\
&= \frac{1}{2m^2} \left\langle \psi^{(0)} \middle| (H_0 - V)^3 \middle| \psi^{(0)} \right\rangle \\
&= \frac{1}{2m^2} \left[(E_n^{(0)})^3 + 3(E_n^{(0)})^2 \langle V \rangle + 3(E_n^{(0)}) \langle V^2 \rangle + \langle V^3 \rangle \right] \\
&= \frac{1}{2m^2} \left[(E_n^{(0)})^3 - 3e^2 (E_n^{(0)})^2 \left\langle \frac{1}{r} \right\rangle + 3e^4 (E_n^{(0)}) \left\langle \frac{1}{r^2} \right\rangle - e^8 \left\langle \frac{1}{r^3} \right\rangle \right] \\
&= \frac{1}{2m^2} \left[(E_n^{(0)})^3 + 6(E_n^{(0)})^3 + 12(E_n^{(0)})^3 \frac{n}{l + \frac{1}{2}} + 8(E_n^{(0)})^3 \frac{n^3}{(l + \frac{1}{2})(l+1)} \right] \\
&= \frac{(E_n^{(0)})^3}{2m^2} \left[7 + \frac{12n}{l + \frac{1}{2}} + \frac{8n^3}{(l + \frac{1}{2})(l+1)} \right],
\end{aligned}
\tag{22.1}
$$

where $\psi^{(0)}$ is the unperturbed wavefunction, $E_n^{(0)} = \frac{e^2}{8\pi a_0 n^2}$ is the unperturbed hydrogen energy level, of which $a_0 = \frac{4\pi}{e^2 m}$ is the Bohr radius.

When $p^2 \ll m^2$, the logarithmic term due to quantum loop effect is less suppressed than the quadratic $\left(\frac{p^2}{m^2}\right)^2$ term. Thus, the quantum loop effect is in fact easier to be measured than the higher order relativistic correction.

22.2

The on-shellness of the spinors means the $\frac{p^\mu p^\nu}{M^2}$ part in the numerator of the propagator dropped out because of the Dirac equation. When $s \ll M$,

$$
\frac{1}{s - M^2} = -\frac{1}{M^2} \left(\frac{1}{1 - \frac{s}{M^2}} \right) \to -\frac{1}{M^2} \left[1 + \frac{s}{M^2} + \left(\frac{s}{M} \right)^2 + \cdots \right]. \tag{22.2}
$$

Thus, the next order in the expansion of Eq. (22.15) of the book is

$$i\mathcal{M} \quad -ig^2\frac{s}{M^4}\bar{v}_2\gamma^\mu u_1\bar{u}_3\gamma^\mu v_4. \tag{22.3}$$

22.3

Chapter 23

The renormalization group

23.1

(a) At 1-loop of QED, the operator contributes a very similar diagram as Eq. (23.42) of the book. We have in dimensional regularization,

$$\begin{aligned} i\mathcal{M} &= Ce_R^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{\bar{u}(p_2)\gamma^\mu(\not{p}_2-\not{k}+m)(\not{p}_2-\not{k})(\not{p}_1-\not{k})(\not{p}_1-\not{k}+m)\gamma^\mu u(p_1)\bar{u}(p_3)v(p_4)}{[(p_1-k)^2-m^2][(p_2-k)^2-m^2]k^2} \\ &= M_0\left(e_R\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{d}{k^2}\right)+\text{finite} \\ &\sim i\mathcal{M}_0\left(\frac{e_R^2}{16\pi^2}\mu^{4-d}(-1)\frac{\Gamma(2-\frac{d}{2})}{(1-\frac{d}{2})}d\right)+\text{finite} \\ &= i\mathcal{M}_0\left(\frac{e_R^2}{4\pi^2}\mu^\varepsilon\frac{1}{\varepsilon}\right)+\text{finite} \end{aligned} \tag{23.1}$$

To cancel the divergence, we can renormalize the operator with $C_RZ_C(\bar{\psi}\not{\partial}\psi)(\bar{\psi}\not{\partial}\psi)$ of which $Z_C = 1+\delta_C$, such that

$$\delta_C = -\frac{e_R^2}{16\pi^2}\frac{4}{\varepsilon}. \tag{23.2}$$

Then,

$$C_RZ_C(\bar{\psi}\not{\partial}\psi)(\bar{\psi}\not{\partial}\psi) = C_R\frac{Z_C}{Z_2}(\bar{\psi}^{(0)}\not{\partial}\psi^{(0)})(\bar{\psi}^{(0)}\not{\partial}\psi^{(0)}) \tag{23.3}$$

Since the coefficient of the bare operator is independent of μ, we have

$$0 = \mu\frac{d}{d\mu}\left(\frac{C_RZ_C}{Z_2}\right) = \frac{C_RZ_C}{Z_2}\left[\frac{\mu}{C_R}\frac{dC_R}{d\mu}+\frac{1}{Z_C}\frac{\partial Z_C}{\partial e_R}\mu\frac{de_R}{d\mu}-\frac{1}{Z_2}\frac{\partial Z_2}{\partial e_R}\mu\frac{de_R}{d\mu}\right]. \tag{23.4}$$

For leading order term,

$$\gamma_C = \frac{\mu}{C_R}\frac{dC_R}{d\mu} = \left(-\frac{\partial Z_C}{\partial e_R}+\frac{\partial Z_2}{\partial e_R}\right) = \left(\frac{8e_R}{16\varepsilon\pi^2}-\frac{4e_R}{16\varepsilon\pi^2}\right)\left(-\frac{\varepsilon}{2}e_R\right) = -\frac{e_R^2}{8\pi^2} = -\frac{\alpha}{2\pi}. \tag{23.5}$$

(b) Solving the differential equation gives us

$$\begin{aligned} C_R(\mu) &= C_R(\mu_0)\exp\left[\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{\gamma_C(\alpha)}{\beta(\alpha)} d\alpha\right] \\ &= C_R(\mu_0)\exp\left[\int_{\alpha(\mu_0)}^{\alpha(\mu)} -\frac{3}{4}\frac{d\alpha}{\alpha}\right] \\ &= C_R(\mu_0)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\frac{3}{4}}. \end{aligned} \tag{23.6}$$

Using Eq. (23.32) of the book, $\Lambda_{\text{QED}} = 10^{286}$ eV $= 10^{277}$ GeV, and setting $\mu_0 = 1$ TeV, we can get

$$C_R(\mu = 1\text{ GeV}) \approx 1.0082. \tag{23.7}$$

23.2

The relevant Lagrangian interaction is the Eq. (23.40) of the book:

$$\mathcal{L}_{\text{4F}} = \frac{G_F}{\sqrt{2}}\bar{\psi}_\mu\gamma^\mu P_L\psi_{\nu_\mu}\bar{\psi}_e\gamma^\mu P_L\psi_{\nu_e} + h.c.. \tag{23.8}$$

The tree-level diagram then is the Eq. (23.39) of the book:

$$i\mathcal{M}_0 = i\frac{G_F}{\sqrt{2}}(\bar{u}_2\gamma^\mu P_L u_1)(\bar{u}_3\gamma^\mu P_L v_4), \tag{23.9}$$

where we use the shorthand $u_i = u(p_i)$ and the momentum indices follow the Eq. (23.42) of the book.

The 1-loop diagram is just like the Eq. (23.42) of the book except that a different Feynman rule applied, which leads to

$$i\mathcal{M} = \frac{G_F}{\sqrt{2}}e_R^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{[\bar{u}_2\gamma^\alpha(\not{p}_2 - \not{k} + m_e)\gamma^\mu P_L v_4][\bar{u}_3\gamma^\mu P_L(\not{p}_1 - \not{k} + m_\mu)\gamma^\alpha u_1]}{[(p_1-k)^2 - m_\mu^2][(p_2-k)^2 - m_e^2]k^2}. \tag{23.10}$$

To extract the counterterm, we can set all the external momenta and masses to zero. Thus,

$$\begin{aligned} \mathcal{M} &= \frac{G_F}{\sqrt{2}}\left(-ie_R^2\mu^{4-d}\right)\int\frac{d^dk}{(2\pi)^d}\frac{[\bar{u}_2\gamma^\alpha\not{k}\gamma^\mu P_L v_4][\bar{u}_3\gamma^\mu P_L\not{k}\gamma^\alpha u_1]}{k^6} + \text{finite} \\ &= \frac{G_F}{\sqrt{2}}\left(-ie_R^2\mu^{4-d}\right)\left(\int\frac{d^dk}{(2\pi)^d}\frac{k^\nu k^\beta}{k^6}\right)[\bar{u}_2\gamma^\alpha\gamma^\nu\gamma^\mu P_L v_4]\left[\bar{u}_3\gamma^\mu P_L\gamma^\beta\gamma^\alpha u_1\right] + \text{finite} \\ &= \frac{G_F}{\sqrt{2}}\left(-ie_R^2\mu^{4-d}\right)\left(\int\frac{d^dk}{(2\pi)^d}\frac{g^{\nu\beta}}{dk^4}\right)[\bar{u}_2\gamma^\alpha\gamma^\nu\gamma^\mu P_L v_4]\left[\bar{u}_3\gamma^\mu\gamma^\beta\gamma^\alpha P_L u_1\right] + \text{finite} \\ &= \frac{G_F}{\sqrt{2}}\left(-ie_R^2\mu^{4-d}\right)\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{dk^4}\right)[\bar{u}_2\gamma^\alpha\gamma^\nu\gamma^\mu P_L v_4]\left[\bar{u}_3\gamma^\mu\gamma^\nu\gamma^\alpha P_L u_1\right] + \text{finite}, \end{aligned} \tag{23.11}$$

where we used the fact that $\left\{\gamma^5, \gamma^\mu\right\} = 0$ such that $P_{L/R}\gamma^\mu = \frac{1\mp\gamma^5}{2}\gamma^\mu = \gamma^\mu\frac{1\pm\gamma^5}{2} = \gamma^\mu P_{R/L}$. Now with the gamma matrices identity:

$$\gamma^\mu\gamma^\nu\gamma^\alpha = g^{\mu\nu}\gamma^\alpha + g^{\nu\alpha}\gamma^\mu - g^{\mu\alpha}\gamma^\nu - i\varepsilon^{\beta\mu\nu\alpha}\gamma^\beta\gamma^5. \tag{23.12}$$

By using the anti-symmetric property of the Levi-Civita symbol, one immediately observes that

$$\gamma^\alpha\gamma^\nu\gamma^\mu = \gamma^\mu\gamma^\nu\gamma^\alpha + 2i\varepsilon^{\beta\mu\nu\alpha}\gamma^\beta\gamma^5. \tag{23.13}$$

Using this gamma matrices identity and the Fierz identity Eq. (11.37), the two spinor factors can be transformed as

$$\begin{aligned}
\left[\bar{u}_2\gamma^\alpha\gamma^\nu\gamma^\mu P_L v_4\right]\left[\bar{u}_3\gamma^\mu\gamma^\nu\gamma^\alpha P_L u_1\right] &= \left[\bar{u}_2\left(\gamma^\mu\gamma^\nu\gamma^\alpha + 2i\varepsilon^{\beta\mu\nu\alpha}\gamma^\beta\gamma^5\right)P_L v_4\right]\left[\bar{u}_3\gamma^\mu\gamma^\nu\gamma^\alpha P_L u_1\right]\\
&= 16\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right] + 2i\varepsilon^{\beta\mu\nu\alpha}\left[\bar{u}_2\gamma^\beta\gamma^5 P_L v_4\right]\left[\bar{u}_3\gamma^\mu\gamma^\nu\gamma^\alpha P_L u_1\right]\\
&= 16\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right] + (2i\varepsilon^{\beta\mu\nu\alpha})(-i\varepsilon^{\rho\mu\nu\alpha})\left[\bar{u}_2\gamma^\beta\gamma^5 P_L v_4\right]\left[\bar{u}_3\gamma^\rho\gamma^5 P_L u_1\right]\\
&= 16\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right] - 12\left[\bar{u}_2\gamma^\beta P_L v_4\right]\left[\bar{u}_3\gamma^\beta P_L u_1\right]\\
&= 16\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right] - 12\left[\bar{u}_2\gamma^\beta P_L u_1\right]\left[\bar{u}_3\gamma^\beta P_L v_4\right]\\
&= 4\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right],
\end{aligned} \tag{23.14}$$

where we used the facts that $\varepsilon^{\beta\mu\nu\alpha}$ is anti-symmetric while the metric tensor $g^{\mu\nu}$ is symmetric in the third line, the contraction of the Levi-Civita symbol $\varepsilon^{\beta\mu\nu\alpha}\varepsilon_{\rho\mu\nu\alpha} = -6\delta_\rho^\beta$ as well as $\gamma^5 P_L = -P_L$ in the fourth line, and another time of Fierz identity Eq. (11.35) in the second to the last line.

Thus,

$$\begin{aligned}
\mathcal{M} &= \frac{G_F}{\sqrt{2}}\left(-4ie_R^2\mu^{4-d}\right)\left(\int\frac{d^dk}{(2\pi)^d}\frac{g^{\nu\beta}}{dk^4}\right)\left[\bar{u}_2\gamma^\mu P_L u_1\right]\left[\bar{u}_3\gamma^\mu P_L v_4\right] + \text{finite}\\
&= \mathcal{M}_0\left(-4ie_R^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{g^{\nu\beta}}{dk^4}\right) + \text{finite}\\
&= \mathcal{M}_0\left(\frac{e_R^2}{8\pi^2}\mu^\varepsilon\frac{1}{\varepsilon}\right) + \text{finite}.
\end{aligned} \tag{23.15}$$

To remove this divergence, renormalizing G by $G = G_R Z_G$, and expanding $Z_G = 1 + \delta_G$, we can the extract the counterterm:

$$\delta_G = -\frac{e_R^2}{16\pi^2}\frac{2}{\varepsilon}. \tag{23.16}$$

Now, notice that $\delta_2 = -\frac{e_R^2}{16\pi^2}\frac{2}{\varepsilon} = \delta_G$ at one-loop level and thus, $Z_2 = Z_G$. Using the fact that neutrino is neutral, $Z_{2\mu} = Z_{2e} = Z_2$:

$$\frac{G_R}{\sqrt{2}}Z_G(\bar{\psi}_\mu\gamma^\mu P_L\psi_{\nu_\mu})(\bar{\psi}_e\gamma^\mu P_L\psi_{\nu_e}) = \frac{G_R}{\sqrt{2}}\frac{Z_G}{Z_2}(\bar{\psi}_\mu^{(0)}\gamma^\mu P_L\psi_{\nu_\mu}^{(0)})(\bar{\psi}_e^{(0)}\gamma^\mu P_L\psi_{\nu_e}^{(0)}). \tag{23.17}$$

Setting up the RGE, we get

$$0 = \mu \frac{d}{d\mu}\left(\frac{G_R Z_G}{Z_2}\right) = \frac{G_R Z_G}{Z_2}\left[\frac{\mu}{G_R}\frac{dG_R}{d\mu} + \frac{1}{Z_G}\frac{\partial Z_G}{\partial e_R}\mu\frac{de_R}{d\mu} - \frac{1}{Z_2}\frac{\partial Z_2}{\partial e_R}\mu\frac{de_R}{d\mu}\right], \tag{23.18}$$

and thus,

$$\gamma_G \equiv \frac{\mu}{G_R}\frac{dG_R}{d\mu} = \left(-\frac{\partial Z_G}{\partial e_R} + \frac{\partial Z_2}{\partial e_R}\right)\beta(e_R) = 0 \tag{23.19}$$

since $Z_G = Z_2$. As a result of the vanishing anomalous dimension, $G_F = G_0$. In the Eq. (23.38) of the book, $A = 0$.

23.3

$$\gamma_m \equiv \frac{\mu}{m_R^2}\frac{d}{d\mu}m_R^2$$

$$\frac{d(m_R^2)}{m_R^2} = \gamma_m(\lambda_R)\frac{d\mu}{\mu} = \gamma_m(\lambda_R)\frac{1}{\mu}\frac{d\mu}{d\lambda_R}d\lambda_R = \frac{\gamma_m(\lambda_R)}{\beta(\lambda_R)}d\lambda_R$$

$$\boxed{m_R^2(\mu) = m_R^2(\mu_0)\exp\left[\int_{\lambda_R(\mu_0)}^{\lambda_R(\mu)}\frac{\gamma_m(\lambda_R)}{\beta(\lambda_R)}d\lambda_R\right]} \tag{23.20}$$

is the general solution.

For small λ_R, we can retain only the leading order dependence of λ_R in $\beta(\lambda_R) = \frac{3\lambda_R^2}{16\pi^2}$ and $\gamma_m = \frac{\lambda_R}{16\pi^2}$ when $\varepsilon \to 0$ such that $\frac{\gamma_m(\lambda_R)}{\beta(\lambda_R)} = \frac{1}{3\lambda_R}$.

Integrate out the beta function to extract the leading scale dependence of λ_R:

$$\beta(\lambda_R) \equiv \mu\frac{d}{d\mu}\lambda_R(\mu) = \frac{3\lambda_R^2}{16\pi^2}$$

$$\frac{d\lambda_R}{d(\lambda_R^2)} = \frac{3}{16\pi^2}\frac{d\mu}{\mu}$$

$$\frac{1}{\lambda_R(\mu)} - \frac{1}{\lambda_R(\mu_0)} = \frac{3}{16\pi^2}\ln\frac{\mu_0}{\mu} \tag{23.21}$$

The general solution then goes to

$$\begin{aligned}
\ln\frac{m_R^2(\mu)}{m_R^2(\mu_0)} &= \int_{\lambda_R(\mu_0)}^{\lambda_R(\mu)}\frac{1}{3\lambda_R}d\lambda_R \\
&= \frac{1}{3}\ln\left(\frac{\lambda_R(\mu)}{\lambda_R(\mu_0)}\right) \\
&= \frac{1}{3}\ln\left(1 + \frac{3\lambda_R(\mu)}{16\pi^2}\ln\frac{\mu}{\mu_0}\right) \\
&\approx \frac{\lambda_R(\mu)}{16\pi^2}\ln\frac{\mu}{\mu_0} \\
&= \gamma_m\ln\frac{\mu}{\mu_0},
\end{aligned} \tag{23.22}$$

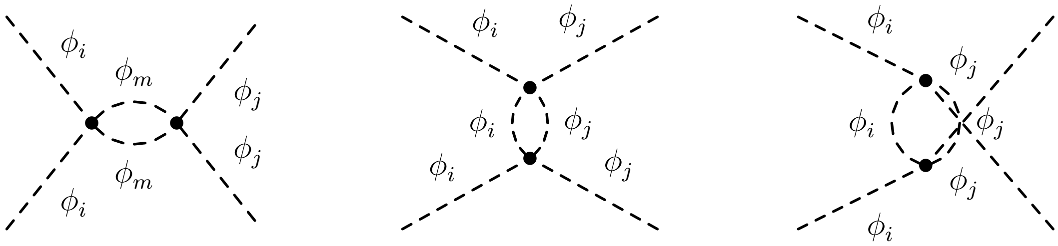
where we used λ_R to be small so that we can expand the ln. Hence, the general solution reduces to

$$\boxed{m_R^2(\mu) = m_R^2(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m}} \tag{23.23}$$

for small λ_R.

23.4

(a) We can use the four-point function $\langle\phi_i(x_1)\phi_i(x_2)\phi_j(x_3)\phi_j(x_4)\rangle$ to renormalize the coupling constant λ. Without loss of generality, I shall assume $i \neq j$. Renormalizing any other four-point functions shall give the same results. Much like the $N = 1$ case, at 1-loop order, we have the diagrams shown in Fig. 23.1. Note that for s-channel, the internal fields ϕ_m can either be $m = i$ or $m = j$ or $m \neq i \neq j$.



(a) s-channel (b) t-channel (c) u-channel

Fig. 23.1: Four-point diagrams at 1-loop order of N-fields ϕ^4 theory.

Since the counterterm δ_λ is all we need, we can set zero external momenta, all of these loops give the same loop integral, but with different multiplicity factors M (expanding in $d = 4 - \varepsilon$ dimensions):

$$= M \times (-i\lambda_R)^2 \mu^{2(4-d)} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2} \frac{i}{k^2} = M \times \mu^{2(4-d)} \frac{\lambda_R^2}{8\pi^2} \frac{i}{\varepsilon}. \tag{23.24}$$

For the multiplicity factors,

- **s-channel:**

$$M = \frac{1}{2!} \times \left(\frac{1}{4}\right)^2 \times [(2\times2\times2\times2)(N-2) + (2\times4!) + (2\times4!)] = \frac{1}{2}(N+4), \tag{23.25}$$

where the factor of $\frac{1}{2!}$ comes from the perturbation expansion, the factor of $\left(\frac{1}{4}\right)^2$ comes from the normalization of couplings, and the rest come from the number of ways of Wick contractions. The term with $N-2$ in the square bracket counts the number of diagram with internal fields ϕ_m differ from neither initial states nor final states. The other two terms are for the internal fields to be the same as either the initial or final states.

- ***t*-channel:**

$$M = \frac{1}{2!} \times \left(\frac{1}{4}\right)^2 \times (2 \times 2 \times 2 \times 2 \times 2) = 1. \tag{23.26}$$

- ***u*-channel:** Same as t-channel:

$$M = 1. \tag{23.27}$$

Overall, $M = \frac{1}{2}(N + 4 + 2 + 2) = \frac{1}{2}(N + 8)$. Hence, the loop integral is

$$= \mu^{2(4-d)} \frac{(N+8)\lambda_R^2}{16\pi^2} \frac{i}{\varepsilon} \tag{23.28}$$

so that

$$\delta_\lambda = \frac{(N+8)\lambda_R}{16\pi^2} \frac{1}{\varepsilon}. \tag{23.29}$$

Since the bare coupling, $\lambda_0 = \mu^{4-d}\lambda_R Z_\lambda$, is μ independent,

$$0 = \mu \frac{d}{d\mu}(\lambda_0) = \mu^\varepsilon \lambda_R Z_\lambda \left(\varepsilon + \frac{\mu}{\lambda_R}\frac{d}{d\mu}\lambda_R + \frac{\mu}{Z_\lambda}\frac{d}{d\mu}\delta_\lambda\right). \tag{23.30}$$

Hence, the β-function to order λ_R^2 is

$$\beta(\lambda_R) \equiv \mu \frac{d}{d\mu}\lambda_R(\mu) = \boxed{-\varepsilon\lambda_R + \frac{(N+8)\lambda_R^2}{16\pi^2}}. \tag{23.31}$$

Similarly, we can extract δ_m from the scalar propagator 1-loop correction. The multiplicity factor is given by

$$M = \frac{1}{4} \times [2 \times (N-1) + 3 \times 2] = \frac{1}{2}(N+2), \tag{23.32}$$

where the term with $N-1$ in square bracket counts the number of diagrams with internal line differ from the initial and final states and the other term counts with the diagrams with internal line the same as the initial and final states. Hence, the leading graph of ϕ_i propagator correction is (expanding in $d = 4 - \varepsilon$ dimensions)

$$\Sigma_2(p^2) = \frac{(N+2)\lambda_R m_R^2}{16\pi^2}\frac{1}{\varepsilon} + \cdots. \tag{23.33}$$

Hence, to $\mathcal{O}(\lambda_R)$,

$$\delta_m = \frac{(N+2)\lambda_R}{16\pi^2}\frac{1}{\varepsilon}. \tag{23.34}$$

As the bare mass $m^2 = m_R^2 Z_m$, and the fact that all fields have the same bare mass in the Lagrangian, we have the RGE:

$$0 = \mu\frac{d}{d\mu}(m^2) = \mu\frac{d}{d\mu}(m_R^2 Z_m) = m_R^2 Z_m \left(\frac{1}{m_R^2}\mu\frac{d}{d\mu}m_R^2 + \frac{1}{Z_m}\mu\frac{d}{d\mu}\delta_m\right) \tag{23.35}$$

and hence

$$\gamma_m \equiv \frac{\mu}{m_R^2}\frac{d}{d\mu}m_R^2 = -\frac{1}{Z_m}\frac{\partial\delta_m}{\partial\lambda_R}\mu\frac{d\lambda_R}{d\mu} = \boxed{\frac{(N+2)\lambda_R}{16\pi^2} + \mathcal{O}(\lambda_R^3)}. \tag{23.36}$$

(b) The Wilson-Fisher fixed point is where the RHS of the RGEs vanish non-trivially:

$$\mu\frac{d}{d\mu}m_R^2 = \frac{(N+2)\lambda_R}{16\pi^2}m_R^2 + \mathcal{O}\big(\lambda_R^2\big), \tag{23.37}$$

$$\mu\frac{d}{d\mu}\lambda_R^2 = -\varepsilon\lambda_R + \frac{(N+8)\lambda_R^2}{16\pi^2} + \mathcal{O}\big(\lambda_R^3\big). \tag{23.38}$$

The location of the Wilson-Fisher fixed point to order ε is

$$\boxed{\lambda_* = \frac{16\pi^2\varepsilon}{N+8}, \qquad m_* = 0}\ . \tag{23.39}$$

(c) At this fixed point, the anomalous dimension is

$$\gamma_m = \frac{(N+2)\varepsilon}{N+8}. \tag{23.40}$$

The critical exponent is

$$\nu = \frac{1}{2-\gamma_m} = \frac{N+8}{2N+16-(N+2)\varepsilon}. \tag{23.41}$$

Doing epsilon expansion in $d=3 \implies \varepsilon = 1$, we have

$$\boxed{\nu = \frac{N+8}{N+14}}\ . \tag{23.42}$$

Note that for $N=1$ (3D Ising model), we reproduce

$$\nu_{N=1} = \frac{9}{15} = 0.6\ . \tag{23.43}$$

For $N=2$ (superfluid transition in ^{4}He), we have

$$\nu_{N=2} = \frac{10}{16} = 0.625\ , \tag{23.44}$$

which is close enough to the theoretical estimate using Monte Carlo simulation including higher order corrections [8]:

$$\nu = 0.6717\ . \tag{23.45}$$

23.5

Chapter 24

Implications of unitarity

24.1

(a) There are four poles in the integrand in Eq. (24.29) in the complex k^0 plane:

$$k_0 = \pm\omega_k \mp i\varepsilon \text{ and } k_0 = p_0 \pm \omega_{k-p} \mp i\varepsilon. \tag{24.1}$$

(b) Closing the contour upward picks up the poles at $k_0 = -\omega_k + i\varepsilon$ and $k_0 = p_0 - \omega_{k-p} + i\varepsilon$. We thus have

$$\begin{aligned}
i\mathcal{M}_{\text{loop}}(p^2) &= \frac{(i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{(k-p)^2 - m^2 + i\varepsilon}\frac{i}{k^2 - m^2 + i\varepsilon} \\
&= \frac{(i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\Bigg[\left(\frac{i}{2\omega_k}\right)\Pi_F(k-p)\left(\frac{1}{k_0 - \omega_k + i\varepsilon} - \frac{1}{k_0 + \omega_k - i\varepsilon}\right) \\
&\quad + \left(\frac{i}{2\omega_{k-p}}\right)\Pi_F(k)\left(\frac{1}{k_0 - p_0 - \omega_{k-p} + i\varepsilon} - \frac{1}{k_0 - p_0 + \omega_{k-p} - i\varepsilon}\right)\Bigg] \\
&= \frac{(i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\Bigg[\left(\frac{i}{2\omega_k}\right)\Pi_F(k-p)(2\pi i)\delta(k_0 + \omega_k) \\
&\quad + \left(\frac{i}{2\omega_{k-p}}\right)\Pi_F(k)(2\pi i)\delta(k_0 - p_0 + \omega_{k-p})\Bigg].
\end{aligned} \tag{24.2}$$

The first delta integrates to 0. Thus, only the second term should contribute.

(c) Since the delta function is real, an imaginary part again can only come from i times the Feynman propagator. Keeping only the second term:

$$\begin{aligned}
2\,\text{Im}\,\mathcal{M}_{\text{loop}}(p^2) &= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{2\omega_{k-p}}\right)\left[(2\pi)\delta(k^2 - m^2)(2\pi)\delta(k_0 - p_0 + \omega_{k-p})\right] \\
&= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2 - m^2)(2\pi)\left[\delta((k-p)^2 - m^2) - \frac{1}{2\omega_{k-p}}\delta(k_0 - p_0 - \omega_{k-p})\right] \\
&= -\frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}(-2\pi i)\delta(k^2 - m^2)(-2\pi i)\delta((k-p)^2 - m^2).
\end{aligned} \tag{24.3}$$

We thus recover Eq. (24.33) of the book. Notice that to get the second line, we used the Eq. (24.32) of the book, and then notice the fact that the second delta function can not be simultaneously satisfied with $\delta(k^2 - m^2)$.

(d) The loop integral is proportional to

$$i\mathcal{M}_{\text{loop}} \sim (i\lambda)^5 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k-p_1)^2 - m^2 + i\varepsilon} \frac{i}{(k-p_1+p_3)^2 - m^2 + i\varepsilon} \times \frac{i}{(k+p_2-p_5)^2 - m^2 + i\varepsilon} \frac{i}{(k+p_2)^2 - m^2 + i\varepsilon}. \tag{24.4}$$

Following previous steps, closing the contour on the upper plane, and noticing that the $\delta(k^0 + \omega_k)$ from $\Pi_F(k)$ and the $\delta(k^0 + p_2^0 + \omega_{k+p_2})$ from $\Pi_F(k+p_2)$ always integrate to 0. Thus, we can expand the loop integral as

$$i\mathcal{M}_{\text{loop}} \sim (i\lambda)^5 \int \frac{d^4k}{(2\pi)^4} \Pi_F(k)\Pi_F(k+p_2) \times \left[\left(-\frac{i}{2\omega_{k-p_1}}\right)(2\pi i)\delta(k^0 - p_1^0 + \omega_{k-p_1})\Pi_F(k-p_1+p_3)\Pi_F(k+p_2-p_5) + \cdots\right], \tag{24.5}$$

where the $\cdots$ contain terms that have each of the three Feynman propagator in the bracket being replaced as a delta function exactly once. Now notice that since this is one-loop, there is only one unknown loop momentum k. For each terms, we can at most replace one more Feynman propagator with a delta function without violating the momentum conservation. This means only one of the four remaining Feynman propagators in each term can be put into on-shell. Further, since the delta function is real, the imaginary part can only come from i times the on-shell Feynman propagator. Using the Eq. (24.32) of the book again reproduce what is expected from the cutting rules.

24.2

Part IV

The Standard Model

Chapter 25

Yang-Mills theory

25.1

From Eq. (25.6) of the book:

$$\mathcal{L}_{\rm YM} = -\frac{1}{4}\sum_a (F^a_{\mu\nu})^2 = -\frac{1}{4}\sum_a (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2, \tag{25.1}$$

and upon the gauge transformation Eq. (25.67) of the book

$$A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) - f^{abc}\alpha^b(x)A^c_\mu(x), \tag{25.2}$$

we observe that

$$\begin{aligned} F^a_{\mu\nu} =& \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\ &\to F^a_{\mu\nu} + \frac{1}{g}\left[\partial_\mu\partial_\nu\alpha^a - \partial_\nu\partial_\mu\alpha^a + \cdots\right] \\ &+ f^{abc}\big[- \partial_\mu(\alpha^b A^c_\nu) + \partial_\nu(\alpha^b A^c_\mu) + (\partial_\mu\alpha^b)A^c_\nu + A^b_\mu(\partial_\nu\alpha^c) + \cdots\big] \\ &+ gf^{abc}\left[-(f^{bde}\alpha^d A^e_\mu)A^c_\nu - A^b_\mu(f^{cde}\alpha^d A^e_\nu) + \cdots\right], \end{aligned} \tag{25.3}$$

where $\cdots$ contain terms that are higher order in α. Notice

- $\partial_\mu\partial_\nu\alpha^a - \partial_\nu\partial_\mu\alpha^a = 0$ simply cancelling out.
- $$\begin{aligned} & f^{abc}\left[-\partial_\mu(\alpha^b A^c_\nu) + \partial_\nu(\alpha^b A^c_\mu) + (\partial_\mu\alpha^b)A^c_\nu + A^b_\mu(\partial_\nu\alpha^c)\right] \\ &= f^{abc}\left[-\partial_\mu(\alpha^b A^c_\nu) + \partial_\nu(\alpha^b A^c_\mu) + (\partial_\mu\alpha^b)A^c_\nu\right] + f^{acb}A^c_\mu(\partial_\nu\alpha^b) \\ &= f^{abc}\left[-\partial_\mu(\alpha^b A^c_\nu) + \partial_\nu(\alpha^b A^c_\mu) + (\partial_\mu\alpha^b)A^c_\nu - A^c_\mu(\partial_\nu\alpha^b)\right] \\ &= f^{abc}\left[-\alpha^b(\partial_\mu A^c_\nu) + \alpha^b(\partial_\nu A^c_\mu)\right] \\ &= -f^{abc}\alpha^b\left(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu\right), \end{aligned}$$

  where we relabel $b \leftrightarrow c$ for the last term to get the second line, and then switching $b \leftrightarrow c$ to get the third line.

-
$$
\begin{aligned}
& gf^{abc}\left[-(f^{bde}\alpha^d A^e_\mu)A^c_\nu - A^b_\mu(f^{cde}\alpha^d A^e_\nu)\right] \\
&= f^{adc}\alpha^b\left(-f^{dbe}A^e_\mu A^c_\nu - A^d_\mu f^{cbe}A^e_\nu\right) \\
&= \alpha^b\left(-f^{aec}f^{ebd}A^d_\mu A^c_\nu - f^{adc}f^{cbe}A^d_\mu A^e_\nu\right) \\
&= \alpha^b A^d_\mu A^e_\nu\left(-f^{ace}f^{cbd} - f^{adc}f^{cbe}\right) \\
&= \alpha^b A^d_\mu A^e_\nu\left(-f^{eac}f^{cbd} - f^{adc}f^{cbe}\right) \\
&= -\alpha^b f^{abc}f^{cde}A^d_\mu A^e_\nu,
\end{aligned}
$$

  where we relabel the dummy variables $b \leftrightarrow d$ to get the second line, and another relabeling of $d \leftrightarrow e$ for the first term to get the third line, and another relabeling of $c \leftrightarrow e$ to get the fourth line. Lastly, we use the Jacobi identity to arrive the last line.

Collecting the uncancelled terms, we then proved Eq. (25.71) of the book about the transformation law for the field strength tensor $F^a_{\mu\nu}$:

$$
F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}. \tag{25.4}
$$

Then, plugging into the Lagrangian,

$$
\mathcal{L}_{\text{YM}} \to= \mathcal{L}_{\text{YM}} - f^{abc}\alpha^b(F^c_{\mu\nu}F^a_{\mu\nu} + F^a_{\mu\nu}F^c_{\mu\nu}) = \mathcal{L}_{\text{YM}}, \tag{25.5}
$$

where the terms vanish because f^{abc} is anti-symmetric with respect to $a \leftrightarrow b$ but the terms inside the bracket is symmetric. Thus, the Yang-Mills Lagrangian is gauge invariant.

25.2

- **Eq. (25.20)**:

$$
\begin{aligned}
T^aT^b &= \frac{1}{2}\left\{T^a, T^b\right\} + \frac{1}{2}\left[T^a, T^b\right] \\
&= \frac{1}{2}\left\{T^a, T^b\right\} + \frac{1}{2}if^{abc}T^c.
\end{aligned} \tag{25.6}
$$

  To get the symmetric part, one can notice that since T^aT^b is well-defined in the fundamental representation, it must be closed. Therefore, it should be an element that can be represented as a linear combination of the identity matrix $\mathbb{I}$ and some generators T^c (in other words, the symmetric part can be separated into a trace part and a traceless part):

$$
\left\{T^a, T^b\right\} = A\mathbb{I} + BT^c \tag{25.7}
$$

  for some group invariant coefficients A and B. The coefficient A can be extracted by taking the trace for both sides:

$$
\frac{1}{2}\delta^{ab} = \text{tr}\left\{[T^aT^b]\right\} = AN, \tag{25.8}
$$

where we used the normalization condition for the fundamental representation Eq. (25.19) of the book. Thus, $A = \frac{1}{2N}\delta^{ab}$. To determine B, we can multiply both sides of Eq. (25.7) by a generator T^d from left and then taking the trace:

$$\operatorname{tr}\Big[T^d\Big\{T^a, T^b\Big\}\Big] = \frac{1}{2N}\delta^{ab}\operatorname{tr}\Big[T^d\Big] + B\operatorname{tr}\Big[T^dT^c\Big] = B\operatorname{tr}\Big[T^dT^c\Big] = \frac{B}{2}\delta^{dc}. \tag{25.9}$$

Thus,

$$\begin{aligned} B &= 2\operatorname{tr}\Big[T^c\Big\{T^a, T^b\Big\}\Big] \\ &= 2\operatorname{tr}\Big[T^cT^aT^b + T^cT^bT^a\Big] \\ &= 2\operatorname{tr}\Big[T^aT^bT^c + T^aT^cT^b\Big] \\ &= 2\operatorname{tr}\Big[T^a\Big\{T^b, T^c\Big\}\Big] \\ &\equiv d^{abc}, \end{aligned} \tag{25.10}$$

where we have used the cyclic property of trace. Thus, we arrive at Eq. (25.20) of the book:

$$T^aT^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2}d^{abc}T^c + \frac{1}{2}if^{abc}T^c. \tag{25.11}$$

- **Eq. (25.21)**:

$$\begin{aligned} \operatorname{tr}\Big[T^aT^bT^c\Big] &= \frac{1}{2}\operatorname{tr}\Big[T^a\Big\{T^b, T^c\Big\}\Big] + \frac{1}{2}\operatorname{tr}\Big[T^a\Big[T^b, T^c\Big]\Big] \\ &= \frac{1}{4}\Big(d^{abc} + 2if^{bcd}\operatorname{tr}\Big[T^aT^d\Big]\Big) \\ &= \frac{1}{4}\Big(d^{abc} + if^{bcd}\delta^{ad}\Big) \\ &= \frac{1}{4}\Big(d^{abc} + if^{bca}\Big) \\ &= \frac{1}{4}\Big(d^{abc} + if^{abc}\Big). \end{aligned} \tag{25.12}$$

- **Eq. (25.22)**:

$$\begin{aligned} \operatorname{tr}\Big[T^aT^bT^cT^d\Big] &= \operatorname{tr}\Big[(T^aT^b)(T^cT^d)\Big] \\ &= \operatorname{tr}\Big[\frac{1}{4N^2}\mathbb{I}^2\delta^{ab}\delta^{cd} + \frac{1}{4}(d^{abe}T^e + if^{abe}T^e)(d^{cdf}T^f + if^{cdf}T^f)\Big] \\ &= \frac{1}{4N}\delta^{ab}\delta^{cd} + \frac{1}{4}(d^{abe} + if^{abe})(d^{cdf} + if^{cdf})\operatorname{tr}\Big[T^eT^f\Big] \\ &= \frac{1}{4N}\delta^{ab}\delta^{cd} + \frac{1}{8}(d^{abe} + if^{abe})(d^{cdf} + if^{cdf})\delta^{ef} \\ &= \frac{1}{4N}\delta^{ab}\delta^{cd} + \frac{1}{8}(d^{abe} + if^{abe})(d^{cde} + if^{cde}), \end{aligned} \tag{25.13}$$

where we plugged in Eq. (25.21) and dropped out the traceless terms to get the second line.

25.3

(a)

(b)

(c)

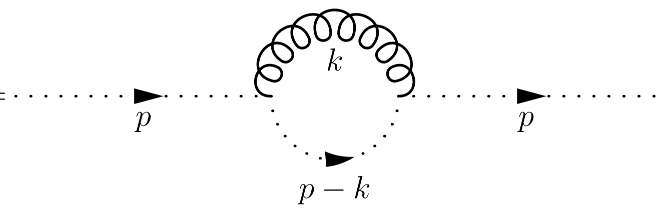
(d)

Chapter 26

Quamtum Yang-Mills theory

26.1

There are only the following ghost self-energy diagram as well as the counterterm diagram contributing the ghost 2-point function at 1-loop:

$i\mathcal{M}^{ab}_{\text{self}} =$ 

$$
\begin{aligned}
&= (-g)^2 f^{dcb} f^{aef} \int \frac{d^4k}{(2\pi)^4} (p^\mu - k^\mu) p^\nu \left[i \frac{-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}}{k^2} \delta^{ce} \right] \frac{i\delta^{df}}{(p-k)^2} \\
&= g^2 C_A \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{-(p-k)\cdot p + (1-\xi)\frac{[(p-k)\cdot k](p\cdot k)}{k^2}}{k^2 (p-k)^2} \\
&= g^2 C_A \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{-(p-k)\cdot p + (1-\xi)\left[\frac{(p\cdot k)^2}{k^2} - p\cdot k\right]}{k^2 (p-k)^2} \\
&= g^2 C_A \delta^{ab} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \left[p^2 \frac{(x-1) - (1-\xi)x}{(k^2-\Delta)^2} + \frac{2(1-x)(1-\xi)[(p\cdot k)^2 + (xp)^2]}{(k^2-\Delta)^3} \right],
\end{aligned}
\tag{26.1}
$$

where $\Delta = -p^2x(1-x)$ in both integrals, and we shift $k \to k + xp$ to get the last line. Also, we dropped the terms linear in k and used Eq. (B.2) of the book. Now notice that

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^d}\frac{(p\cdot k)^2}{(k^2-\Delta)^3} &= \int \frac{d^dk}{(2\pi)^d}\frac{p^\mu k^\nu g^{\mu\nu} p^\alpha k^\beta g^{\alpha\beta}}{(k^2-\Delta)^3} \\ &= \frac{1}{d}\int \frac{d^dk}{(2\pi)^d}\frac{k^2 g^{\nu\beta} g^{\mu\nu} g^{\alpha\beta} p^\mu p^\alpha}{(k^2-\Delta)^3} \\ &= \frac{1}{d}\int \frac{d^dk}{(2\pi)^d}\frac{p^2}{(k^2-\Delta)^3}.\end{aligned} \tag{26.2}$$

Thus, in dimensional regularization,

$$i\mathcal{M}^{ab}_{\text{self}} = g^2 C_A \delta^{ab}\mu^{4-d}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}p^2\left[\frac{(x-1)-(1-\xi)x}{(k^2-\Delta)^2} + \frac{2(1-x)(1-\xi)(\frac{1}{d}k^2+x^2)}{(k^2-\Delta)^3}\right]. \tag{26.3}$$

To extract the counterterm, we only need the divergent part:

$$\begin{aligned}\mathcal{M}^{ab}_{\text{self}} &= g^2 C_A\delta^{ab}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)p^2\left[(x-1)+\frac{1}{2}(1-3x)(1-\xi)\right] + \text{finite} \\ &= C_A\delta^{ab}\left(\frac{g^2}{16\pi^2}\right)p^2\left(-1-\frac{1}{2}(1-\xi)\right)\frac{1}{\varepsilon} + \text{finite}.\end{aligned} \tag{26.4}$$

In $\overline{\text{MS}}$ scheme, adding the counterterm contribution

$$i\mathcal{M}^{ab}_{\text{c.t.}} = \quad \text{<image>} \quad = ip^2\delta^{ab}\delta_{3c} \tag{26.5}$$

shall just remove the divergent part. Therefore, one must choose

$$\delta_{3c} = \frac{1}{\varepsilon}\left(\frac{g^2}{16\pi^2}\right)\left[C_A + \frac{1}{2}(1-\xi)C_A\right], \tag{26.6}$$

which is exactly Eq. (26.84) of the book as expected.

26.2

Chapter 27

Gluon scattering and the spinor-helicity formalism

27.1

- **I believe there is a typo in the question: by definition, the reference momentum r^μ must not be aligned with p^μ, so the reference momentum can not be $(1, 0, 0, 1)$. I will try $r^\mu = (1, 0, 0, -1)$ instead.**

  By Eq. (27.12) of the book, given that $p^\mu = (E, 0, 0, E)$ and $r^\mu = (1, 0, 0, -1)$, we have

$$p^{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}, \tag{27.1}$$

  and

$$r^{\beta\dot{\beta}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{27.2}$$

  From Eq. (27.15) of the book, these are outer product of spinors:

$$p^{\alpha\dot{\alpha}} = \lambda_p^\alpha \tilde{\lambda}_p^{\dot{\alpha}}, \tag{27.3}$$

  and

$$r^{\beta\dot{\beta}} = \lambda_r^\beta \tilde{\lambda}_r^{\dot{\beta}}. \tag{27.4}$$

  We can infer that

$$\lambda_p^\alpha = \begin{pmatrix} 0 \\ a_{p_2} \end{pmatrix}, \ \tilde{\lambda}_p^{\dot{\alpha}} = \begin{pmatrix} 0 & b_{p_2} \end{pmatrix}, \ \text{with } a_{p_2} b_{p_2} = 2E, \tag{27.5}$$

  and

$$\lambda_r^\beta = \begin{pmatrix} a_{r_1} \\ 0 \end{pmatrix}, \ \tilde{\lambda}_r^{\dot{\beta}} = \begin{pmatrix} b_{r_1} & 0 \end{pmatrix}, \ \text{with } a_{r_1} b_{r_1} = 2. \tag{27.6}$$

  For real momenta, where $\lambda^\alpha = (\tilde{\lambda}^{\dot{\alpha}})^\dagger$, we have $a_{p_2} = b_{p_2} = \sqrt{2E}$ and $a_{r_1} = b_{r_1} = \sqrt{2}$. Thus,

$$p\rangle = \lambda_p^\alpha = \begin{pmatrix} 0 \\ \sqrt{2E} \end{pmatrix}, \ \text{and } p] = \tilde{\lambda}_p^{\dot{\alpha}} = \begin{pmatrix} 0 & \sqrt{2E} \end{pmatrix}, \tag{27.7}$$

and

$$r\rangle = \lambda_r^{\beta} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, \ r] = \tilde{\lambda}_r^{\dot{\beta}} = \begin{pmatrix} \sqrt{2} & 0 \end{pmatrix}. \tag{27.8}$$

By using Eq. (27.20) of the book with zero phase, we also have

$$[pr] = \langle rp\rangle = 2\sqrt{E}. \tag{27.9}$$

Plugging these into Eq. (27.29) of the book, we get

$$[\epsilon_p^-(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{p\rangle[r}{[pr]} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \tag{27.10}$$

and

$$[\epsilon_p^+(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{r\rangle[p}{\langle rp\rangle} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \tag{27.11}$$

For each μ, we multiply the μth Pauli matrix with the polarization bispinor and taking the trace:

$$\epsilon_\pm^\mu = \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}\epsilon_\pm^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}}(0,1,\pm i,0). \tag{27.12}$$

- Notice that Eq. (27.29) is only applicable for lightlike polarization, which is not the case. However, one can observe that $\epsilon^\mu = \frac{1}{\sqrt{2}}(\epsilon_+^\mu + \epsilon_-^\mu)$. Thus, the reference momentum r^μ can still be taken as $r^\mu = (1,0,0,-1)$.

27.2

Chapter 28

Spontaneous symmetry breaking

28.1

Plugging $\phi(x) = \sqrt{\frac{2m^2}{\lambda}} + \tilde{\phi}(x) = v + \tilde{\phi}(x)$ back into the Lagrangian Eq. (28.10) of the book,

$$\begin{aligned} \mathcal{L} = &(\partial_\mu \tilde{\phi}^*)(\partial_\mu \tilde{\phi}) + m^2 \left[v^2 + \tilde{\phi}^* \tilde{\phi} + v \left(\tilde{\phi}^* + \tilde{\phi} \right) \right] \\ &- \frac{\lambda}{4} \left[v^4 + \left(\tilde{\phi}^* \tilde{\phi} \right)^2 + v^2 \left(\tilde{\phi}^* + \tilde{\phi} \right)^2 + 2v^2 \tilde{\phi}^* \tilde{\phi} + 2v^3 \left(\tilde{\phi}^* + \tilde{\phi} \right) + 2v \left(\tilde{\phi}^* \tilde{\phi} \right) \left(\tilde{\phi}^* + \tilde{\phi} \right) \right]. \end{aligned} \tag{28.1}$$

Collecting the bilinear term $\tilde{\phi}$ and $\tilde{\phi}^*$ in potential to write out the mass square matrix leads to

$$M^2 = \begin{pmatrix} \frac{\lambda}{4} v^2 & -\frac{m^2}{2} + \frac{\lambda}{4} v^2 + \frac{\lambda}{4} v^2 \\ -\frac{m^2}{2} + \frac{\lambda}{4} v^2 + \frac{\lambda}{4} v^2 & \frac{\lambda}{4} v^2 \end{pmatrix} = \frac{m^2}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{28.2}$$

where we have used $\lambda v^2 = 2m^2$. Solving for the two eigenvalues m_1^2 and m_2^2 by noticing that

$$m_1^2 m_2^2 = \det\left(M^2\right) = 0, \tag{28.3}$$

and

$$m_1^2 + m_2^2 = \mathrm{Tr}(M) = m^2. \tag{28.4}$$

The two eigenvalues are thus

$$m_1^2 = 0, \ m_2^2 = m^2, \tag{28.5}$$

or

$$m_1 = 0, \ m_2 = m. \tag{28.6}$$

Indeed, the mass matrix has a zero eigenvalue. Solving for the two eigenvectors $\vec{\phi}_1$ and $\vec{\phi}_2$:

$$\vec{\phi}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \vec{\phi}_2 = -\frac{i}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{28.7}$$

Therefore, the linear combination of the complex field $\tilde{\phi}$ that diagonalize the mass matrix are just $\phi_1 = \frac{1}{2}\left(\tilde{\phi} + \tilde{\phi}^*\right)$ and $\phi_2 = -\frac{i}{2}\left(\tilde{\phi} - \tilde{\phi}^*\right)$, which are just the real and imaginary degree of freedom of $\tilde{\phi}$, respectively. Writing $\tilde{\phi} = \phi_1 + i\phi_2$ or $\phi(x) = v + \phi_1(x) + i\phi_2(x)$ and plugging this back to Eq. (28.10) of the book, we can get

$$\begin{aligned}
\mathcal{L} &= (\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + m^2\left[(v+\phi_1)^2 + \phi_2^2\right] - \frac{\lambda}{4}\left[(v+\phi_1)^4 + \phi_2^4 + 2(v+\phi_1)^2\phi_2^2\right] \\
&= (\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + m^2v^2 + 2m^2v\phi_1 + m^2\phi_1^2 + m^2\phi_2^2 \\
&\quad - \frac{\lambda}{4}v^4 - \frac{\lambda}{4}\phi_1^4 - \lambda v\phi_1^3 - \lambda v^3\phi_1 - \frac{3}{2}v^2\phi_1^2 - \frac{\lambda}{4}\phi_2^4 - \frac{\lambda}{2}v^2\phi_2^2 - \frac{\lambda}{2}\phi_1^2\phi_2^2 - \lambda v\phi_1\phi_2^2 \\
&= (\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + \frac{m^4}{\lambda} - 2m^2\phi_1^2 - \frac{\lambda}{4}\phi_1^4 - \sqrt{2\lambda}m\phi_1^3 - \frac{\lambda}{4}\phi_2^4 - \frac{\lambda}{2}\phi_1^2\phi_2^2 - m\sqrt{2\lambda}\phi_1\phi_2^2.
\end{aligned} \tag{28.8}$$

To see how this is related to Eq. (28.12), one need to note that this $\tilde{\phi}(x)$ is expanding around a purely real VEV (unlike Eq. (28.11) of the book). The two descriptions thus should coincide at where $\frac{\pi(x)}{F_\pi} = 0$ such that one can associate

$$v + \phi_1(x) = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)\cos\frac{\pi(x)}{F_\pi} = v + \frac{1}{\sqrt{2}}\sigma(x), \text{ or } \phi_1(x) = \frac{1}{\sqrt{2}}\sigma(x), \tag{28.9}$$

and

$$\phi_2(x) = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)\sin\frac{\pi(x)}{F_\pi} = 0. \tag{28.10}$$

The only non-vanishing term involving ϕ_2 in Eq. (28.8) is

$$(\partial_\mu\phi_2)^2 = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2\cos^2\frac{\pi(x)}{F_\pi} = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2. \tag{28.11}$$

Plugging these back into Eq. (28.8),

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2 - \left(-\frac{m^4}{\lambda} + m^2\sigma^2 + \frac{1}{2}\sqrt{\lambda}m\sigma^3 + \frac{1}{16}\lambda\sigma^4\right), \tag{28.12}$$

which is exactly Eq. (28.12) of the book.

28.2

Chapter 29

Weak interaction

29.1

At tree-level, there is only one s-channel diagram mediating by a Z boson for this process. From Eq. (29.40)-(29.44) of textbook, the Z couples to the fermion currents J_μ^Z as $\mathcal{L} = \frac{e}{\sin\theta_w} Z_\mu J_\mu^Z$, where

$$\begin{aligned} J_\mu^Z &= \frac{1}{\cos\theta_w}(J_\mu^3 - \sin^2\theta_w J_\mu^{\text{EM}}) \\ &= \frac{1}{\cos\theta_w}\left[\left(T^3 - Q\sin^2\theta_w\right)\bar{\psi}_L\gamma^\mu\psi_L - Q\sin^2\theta_w\bar{\psi}_R\gamma^\mu\psi_R\right] \\ &= \frac{1}{\cos\theta_w}\left[\left(T^3 - Q\sin^2\theta_w\right)\bar{\psi}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi - Q\sin^2\theta_w\bar{\psi}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi\right] \\ &= \frac{1}{\cos\theta_w}\left[\frac{1}{2}\left(T^3 - 2Q\sin^2\theta_w\right)\bar{\psi}\gamma^\mu\psi - \frac{1}{2}T^3\bar{\psi}\gamma^\mu\gamma^5\psi\right], \end{aligned} \tag{29.1}$$

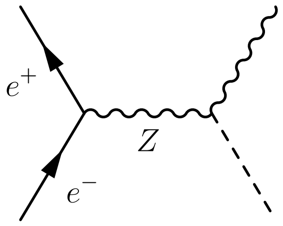
We have separated the current into a vector part and an axial-vector part by inserting the γ_5 matrices using $P_{L/R} = \frac{1\mp\gamma_5}{2}$.

Since the electron carries an electric charge $Q = -1$, and the left-handed electron also carries a weak isospin $T^3 = -\frac{1}{2}$ while the right-handed is weak isospin neutral, we thus have

$$J_\mu^Z = \frac{1}{\cos\theta_w}\left[\left(-\frac{1}{4} + \sin^2\theta_w\right)\bar{e}\gamma^\mu e + \frac{1}{4}\bar{e}\gamma^\mu\gamma^5 e\right]. \tag{29.2}$$

We then calculate the amplitude under unitary gauge

$i\mathcal{M} =$



$$
\begin{aligned}
&= \left(\frac{ie}{\sin\theta_w\cos\theta_w}\right)^2\left(\frac{m_W}{\cos\theta_w}\right)\left[\left(-\frac{1}{4}+\sin^2\theta_w\right)\bar{e}\gamma^\mu e+\frac{1}{4}\bar{e}\gamma^\mu\gamma^5 e\right]\frac{-i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{m_Z^2}\right)}{s-m_Z^2}g^{\nu\alpha}\epsilon^{*\alpha}\\
&= i\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^2\left(\frac{m_W}{\cos\theta_w}\right)\left[\left(-\frac{1}{4}+\sin^2\theta_w\right)\bar{e}\gamma^\mu e+\frac{1}{4}\bar{e}\gamma^\mu\gamma^5 e\right]\frac{1}{s-m_Z^2}\epsilon^{*\mu}.
\end{aligned}
\tag{29.3}
$$

Notice the $\frac{p^\mu p^\nu}{m_Z^2}$ part in the propagator, when coupling with the vector current part does not contribute since for the on-shell initial electrons, we can use the Dirac equation and $p^\mu = p_1^\mu + p_2^\mu$ to see

$$
\bar{v}(p_1)\gamma^\mu u(p_2)p^\mu p^\nu = \bar{v}(p_1)(\not{p}_1+\not{p}_2)u(p_2)p^\nu = \bar{v}(p_1)(-m_e+m_e)u(p_2)p^\nu = 0. \tag{29.4}
$$

However, for the axial-vector current, this does not vanish because

$$
\begin{aligned}
\bar{v}(p_1)\gamma^\mu\gamma^5 u(p_2)p^\mu p^\nu &= \bar{v}(p_1)(\not{p}_1+\not{p}_2)\gamma^5 u(p_2)p^\nu\\
&= \bar{v}(p_1)\not{p}_1\gamma^5 u(p_2)p^\nu - \bar{v}(p_1)\gamma^5\not{p}_2 u(p_2)p^\nu\\
&= \bar{v}(p_1)(-m_e-m_e)u(p_2)p^\nu\\
&= -2m_e\bar{v}(p_1)u(p_2)p^\nu,
\end{aligned}
\tag{29.5}
$$

which is generally nonzero unless $m_e = 0$. This is in fact related to chiral anomaly[1]. Another way to see the non-vanishing nature of this term is by Goldstone boson equivalence theorem. The $\frac{p^\mu p^\nu}{m_Z^2}$ part in the propagator comes from the longitudinal polarization of the massive Z boson, whose contribution is equivalent to a Goldstone gauge boson and the pseudoscalar current is exactly one would expect from the interaction with a Goldstone boson. However, as for LEP, $s = (206\text{ GeV})^2 \gg m_e^2 \approx (0.511\text{ MeV})^2$, we can safely set $m_e = 0$ to do above calculation and ignore the contribution of this term, which is why we have simplified the propagator in the last

[1] In fact, this is anomalous at quantum level even for massless fermions, as will be explored in the next chapter.

line of Eq. (29.3). The amplitude square is then

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= -\frac{1}{4}\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4\left(\frac{m_W}{\cos\theta_w}\right)^2\left(\frac{1}{s-m_Z^2}\right)^2 \\ &\qquad\times\left\{\left(-\frac{1}{4}+\sin^2\theta_w\right)^2\mathrm{Tr}\Big[\not{p}_1\gamma^\mu\not{p}_2\gamma^\mu\Big]+\frac{1}{16}\mathrm{Tr}\Big[\not{p}_1\gamma^\mu\gamma^5\not{p}_2\gamma^\mu\gamma^5\Big]\right\} \\ &= \left(\frac{1}{4}-\sin^2\theta_w+2\sin^4\theta_w\right)\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4\left(\frac{m_Z}{s-m_Z^2}\right)^2(p_1\cdot p_2) \\ &= \left(\frac{1}{8}-\frac{1}{2}\sin^2\theta_w+\sin^4\theta_w\right)\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4\left(\frac{m_Z}{s-m_Z^2}\right)^2 s \\ &= \left(2-8\sin^2\theta_w+16\sin^4\theta_w\right)\frac{E_{\text{CM}}^2 m_Z^6}{v^4(E_{\text{CM}}^2-M_Z^2)^2}\end{aligned} \tag{29.6}$$

where we used the trick Eq. (13.112) of the textbook that in any physical matrix element, one can do the replacement $\sum_{\text{pols. i}}\epsilon_\mu^{i*}\epsilon_\nu^i \to -g_{\mu\nu}$ for external polarization to get the first line. Also notice that there can not be any current and axial-current cross product contribution in the trace terms because any such terms have either Tr[odd # of γ-matrices] $=0$ or $\mathrm{Tr}\big[\gamma^5\times \text{odd \# of } \gamma^\mu\big] = -\mathrm{Tr}\big[\text{odd \# of } \gamma^\mu\times\gamma^5\big] = -\mathrm{Tr}\big[\gamma^5\times\text{odd \# of }\gamma^\mu\big] = 0$. Then, we used $v=\frac{2m_W\sin\theta_w}{e}=\frac{2m_Z\sin\theta_w\cos\theta_w}{e}$ to get the final answer.

In CM frame, we have

$$p_i = |\vec{p}_1| = |\vec{p}_2| = \frac{E_{\text{CM}}}{2}. \tag{29.7}$$

To solve for the $p_f = |\vec{p}_h| = |\vec{p}_Z|$, we used the relations

$$p_f^2 = E_h^2 - m_h^2 = E_Z^2 - m_Z^2, \tag{29.8}$$

and

$$E_{\text{CM}} = E_h + E_Z, \tag{29.9}$$

to get

$$E_h = \frac{E_{\text{CM}}^2 - m_Z^2 + m_h^2}{2E_{\text{CM}}}, \tag{29.10}$$

or

$$\frac{p_f}{p_i} = 2\sqrt{\frac{E_h^2-m_h^2}{E_{\text{CM}}^2}} = 2\sqrt{\frac{\left(E_{\text{CM}}^2-m_Z^2+m_h^2\right)^2}{4E_{\text{CM}}^4}-\frac{m_h^2}{E_{\text{CM}}^2}}. \tag{29.11}$$

From Eq. (5.32) of textbook, we have

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{64\pi^2E_{\text{CM}}^2}\frac{p_f}{p_i}|\mathcal{M}|^2 \\ &= \frac{\left(2-8\sin^2\theta_w+16\sin^4\theta_w\right)m_Z^6}{64\pi^2v^4(E_{\text{CM}}^2-M_Z^2)^2}\frac{p_f}{p_i}.\end{aligned} \tag{29.12}$$

Integrating over solid angle and plugging the values $E_{\text{CM}} = 206$ GeV, $\sin^2\theta_w = 0.223$, $v = 247$ GeV, $m_Z = 91.7816$ GeV, and $m_h = 100$ GeV (of course, the actual measured value of the mass of Higgs boson is about 125 GeV, but if we take the actual value, the LEP $E_{\text{CM}} =$ 206 GeV $< m_Z + m_h$ is not enough to produce this process resonantly) leads us to the cross section

$$\sigma = \frac{\left(2 - 8\sin^2\theta_w + 16\sin^4\theta_w\right) m_Z^6}{16\pi v^4 (E_{\text{CM}}^2 - M_Z^2)^2}\frac{p_f}{p_i} \approx 1.0192\times 10^{-9}\ (\text{GeV})^{-2} \approx 0.3969\ \text{pb}. \tag{29.13}$$

As a sanity check, we used MadGraph5 to generate this process and get a cross section of 0.4263 pb [9], which is not bad.

29.2

(a) Using Eq. (29.40) from textbook, we can get the Z-transmitted diagram under unitary gauge as

$$\begin{aligned} i\mathcal{M}_Z &= \left(\frac{ie}{\sin\theta_w}\right)^2 J_\mu^{Zee}\frac{-i\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m_Z^2}\right)}{s - m_Z^2}J_\nu^{Zqq} \\ &= i\left(\frac{e}{\sin\theta_w}\right)^2 J_\mu^{Zee}\frac{1}{s - m_Z^2}J_\mu^{Zqq}, \end{aligned} \tag{29.14}$$

where J_μ^Z is given by Eq. (29.1), and we again ignore the $\frac{p^\mu p^\nu}{m_Z^2}$ part in the propagator because the mass of all flavors of quarks except for the top's are much smaller than m_Z and we will ignore the quark mass (as well as electron mass) from here on, while the top is too heavy to hadronize before it decays and thus, is not related to this question. For clear notation purpose, I shall define a "Z charge" Q_z for each particle coupling to the Z boson as

$$Q_z \equiv T^3 - Q\sin^2\theta_w. \tag{29.15}$$

Notice that the left-handed and right-handed of the same flavor particle do not share the same weak charge, unlike the electric charge Q. Then, we can write the Z-boson transmitted current as

$$J_\mu^Z = \frac{1}{2\cos\theta_w}\left[(Q_{zR} + Q_{zL})\bar{\psi}\gamma^\mu\psi + (Q_{zR} - Q_{zL})\bar{\psi}\gamma^\mu\gamma^5\psi\right], \tag{29.16}$$

For the γ-transmitted diagram under Feynman gauge, we have

$$i\mathcal{M}_\gamma = ie^2 J_\mu^{\gamma ee}\frac{1}{s}J_\mu^{\gamma qq}, \tag{29.17}$$

where J_μ^γ is given by Eq. (29.44) of textbook:

$$J_\mu^\gamma = Q\bar{\psi}\gamma^\mu\psi. \tag{29.18}$$

For later convenience, let's write out the some trace expression and their products:

$$\text{Tr}\left[\not{p}_i\gamma^\mu\not{p}_j\gamma^\nu\right] = \text{Tr}\left[\not{p}_i\gamma^\mu\gamma^5\not{p}_j\gamma^\nu\gamma^5\right] = 4(p_i^\mu p_j^\nu - p_{ij}g^{\mu\nu} + p_i^\nu p_j^\mu), \tag{29.19}$$

where $p_{ij} \equiv p_i \cdot p_j$, and

$$\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \gamma^5 \not{p}_j \gamma^\nu\Big] = \mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu \gamma^5\Big] = -4i\varepsilon^{\mu\nu\alpha\beta} p_i^\alpha p_j^\beta. \tag{29.20}$$

Then,

$$\mathrm{Tr}\Big[\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu\Big] \mathrm{Tr}\Big[\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu\Big] = 32(p_{13}p_{24} + p_{14}p_{23}) = 8(t^2 + u^2), \tag{29.21}$$

and

$$\begin{aligned}
\mathrm{Tr}\Big[\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu \gamma^5\Big] \mathrm{Tr}\Big[\not{p}_3 \gamma^\mu \not{p}_4 \gamma^\nu \gamma^5\Big] &= -16(\varepsilon^{\nu\mu\alpha\beta} p_1^\alpha p_2^\beta)(\varepsilon^{\mu\nu\rho\delta} p_3^\rho p_4^\delta) \\
&= 16(\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\rho\delta}) p_1^\alpha p_2^\beta p_3^\rho p_4^\delta \\
&= 32(g^{\alpha\rho}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\rho}) p_1^\alpha p_2^\beta p_3^\rho p_4^\delta \\
&= 32(p_{13}p_{24} - p_{14}p_{23}) \\
&= 8(t^2 - u^2).
\end{aligned} \tag{29.22}$$

All other trace products can either be converted to the forms above or vanish. This can be seen from the fact that Eq. (29.19) is symmetric w.r.t. $\mu \leftrightarrow \nu$ while Eq. (29.20) is anti-symmetric w.r.t. $\mu \leftrightarrow \nu$ so their product must vanish. Then, the spin sum of currents can be written out as

$$\begin{aligned}
\sum_{\text{spins}} (J_\mu^Z)(J_\nu^Z)^\dagger = \left(\frac{1}{2\cos\theta_w}\right)^2 \Big\{ &(Q_{zR} + Q_{zL})^2 \,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu\Big] + (Q_{zR} - Q_{zL})^2 \,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \gamma^5 \not{p}_j \gamma^\nu \gamma^5\Big] \\
&+ (Q_{zR}^2 - Q_{zL}^2)\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu \gamma^5\Big] + (Q_{zR}^2 - Q_{zL}^2)\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \gamma^5 \not{p}_j \gamma^\nu\Big] \Big\} \\
= \left(\frac{1}{\sqrt{2}\cos\theta_w}\right)^2 \Big\{ &(Q_{zR}^2 + Q_{zL}^2)\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu\Big] + (Q_{zR}^2 - Q_{zL}^2)\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu \gamma^5\Big] \Big\},
\end{aligned} \tag{29.23}$$

$$\sum_{\text{spins}} (J_\mu^\gamma)(J_\nu^\gamma)^\dagger = Q^2\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu\Big], \tag{29.24}$$

and

$$\begin{aligned}
\sum_{\text{spins}} (J_\mu^Z)(J_\nu^\gamma)^\dagger &= \sum_{\text{spins}} (J_\mu^\gamma)(J_\nu^Z)^\dagger \\
&= \left(\frac{Q}{2\cos\theta_w}\right) \Big\{ (Q_{zR} + Q_{zL})\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu\Big] + (Q_{zR} - Q_{zL})\,\mathrm{Tr}\Big[\not{p}_i \gamma^\mu \not{p}_j \gamma^\nu \gamma^5\Big] \Big\}.
\end{aligned} \tag{29.25}$$

Also, notice that in the massless limit in the CM frame,

$$t = -2p_{13} = -2E^2(1 - \cos\theta) = -\frac{s}{2}(1 - \cos\theta), \tag{29.26}$$

and

$$u = -2p_{14} = -2E^2(1 + \cos\theta) = -\frac{s}{2}(1 + \cos\theta), \tag{29.27}$$

where $E = \sqrt{s}/2$ is the energy of individual particle, and θ is the scattering angle between the incoming particle and the outgoing particle. Then,

$$t^2 + u^2 = \frac{s^2}{2}(1 + \cos^2\theta), \tag{29.28}$$

and

$$t^2 - u^2 = -s^2\cos\theta. \tag{29.29}$$

Using results above, the Z contribution along is

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_Z|^2 &= \frac{3}{4}\left(\frac{e}{\sin\theta_w}\right)^4\left(\frac{1}{s - m_Z^2}\right)^2\sum_s\sum_{s'}(J_\mu^{Zee})(J_\nu^{Zee})^\dagger(J_\mu^{Zqq})(J_\nu^{Zqq})^\dagger \\
&= \frac{3}{2}\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4\left(\frac{1}{s - m_Z^2}\right)^2 \\
&\quad\times\left\{(Q_{zR,e}^2 + Q_{zL,e}^2)(Q_{zR,q}^2 + Q_{zL,q}^2)(t^2 + u^2) + (Q_{zR,e}^2 - Q_{zL,e}^2)(Q_{zR,q}^2 - Q_{zL,q}^2)(t^2 - u^2)\right\} \\
&= \frac{3}{4}\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4\left(\frac{s}{s - m_Z^2}\right)^2 \\
&\quad\times\left\{(Q_{zR,e}^2 + Q_{zL,e}^2)(Q_{zR,q}^2 + Q_{zL,q}^2)(1 + \cos^2\theta) - 2(Q_{zR,e}^2 - Q_{zL,e}^2)(Q_{zR,q}^2 - Q_{zL,q}^2)\cos\theta\right\},
\end{aligned} \tag{29.30}$$

where a factor of 3 comes from the color sum of quarks. Similarly, the γ contribution along is

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_\gamma|^2 = \frac{6Q_e^2Q_q^2e^4}{s^2}(t^2 + u^2) = 3Q_e^2Q_q^2e^4(1 + \cos^2\theta), \tag{29.31}$$

and the interference between the two diagrams is

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}\left(\mathcal{M}_Z\mathcal{M}_\gamma^\dagger + \mathcal{M}_\gamma\mathcal{M}_Z^\dagger\right) &= \frac{3}{4}\frac{e^4}{\sin^2\theta_w}\frac{1}{s(s - m_Z^2)}2\sum_s\sum_{s'}(J_\mu^{Zee})(J_\nu^{\gamma ee})^\dagger(J_\mu^{Zqq})(J_\nu^{\gamma qq})^\dagger \\
&= \frac{3}{2}\frac{Q_eQ_qe^4}{\sin^2\theta_w\cos^2\theta_w}\frac{2}{s(s - m_Z^2)} \\
&\quad\times\left\{(Q_{zR,e} + Q_{zL,e})(Q_{zR,q} + Q_{zL,q})(t^2 + u^2) + (Q_{zR,e} - Q_{zL,e})(Q_{zR,q} - Q_{zL,q})(t^2 - u^2)\right\} \\
&= \frac{3}{4}\frac{Q_eQ_qe^4}{\sin^2\theta_w\cos^2\theta_w}\frac{2s^2}{s(s - m_Z^2)} \\
&\quad\times\left\{(Q_{zR,e} + Q_{zL,e})(Q_{zR,q} + Q_{zL,q})(1 + \cos^2\theta) - 2(Q_{zR,e} - Q_{zL,e})(Q_{zR,q} - Q_{zL,q})\cos\theta\right\}
\end{aligned} \tag{29.32}$$

Particle	Q	Q_{zL}	Q_{zR}
e^-	-1	$-\frac{1}{2}+\sin^2\theta_w$	$\sin^2\theta_w$
$u,\ c$	$+\frac{2}{3}$	$\frac{1}{2}-\frac{2}{3}\sin^2\theta_w$	$-\frac{2}{3}\sin^2\theta_w$
$d,\ s,\ b$	$-\frac{1}{3}$	$-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w$	$\frac{1}{3}\sin^2\theta_w$

Table 29.1: Electric charge and "Z charge" for the electron and quarks.

electric charge Q and "Z charge" $Q_z \equiv T^3 - Q\sin^2\theta_w$ for each fermions relevant to this problem is listed in Table 29.1.

Before calculating the cross sections, one should notice that since Z boson is heavy and unstable, one should really really replace the usual Z propagator above by the Breit-Wigner modified propagator from Eq. (24.50) of textbook (otherwise, the plot of cross section will also be divergent at $s = m_Z^2$):

$$iG(s) = \frac{i}{s - m_Z^2 + im_Z\Gamma_Z} \tag{29.33}$$

Now we know that the differential cross section is given by

$$\left(\frac{d\sigma}{d(\cos\theta)}\right)_{\text{CM}} = \frac{1}{32\pi s}|\mathcal{M}|^2 \tag{29.34}$$

The cross sections are

$$\begin{aligned}\sigma_Z(e^+e^- \to \bar{q}q) &= \frac{s}{64\pi}\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4 \left|\frac{1}{s - m_Z^2 + im_Z\Gamma_Z}\right|^2 \\ &\quad \times \left\{4(Q_{zR,e}^2 + Q_{zL,e}^2)(Q_{zR,q}^2 + Q_{zL,q}^2) - 3(Q_{zR,e}^2 - Q_{zL,e}^2)(Q_{zR,q}^2 - Q_{zL,q}^2)\right\} \\ &= \frac{s}{64\pi}\left(\frac{e}{\sin\theta_w\cos\theta_w}\right)^4 \frac{1}{(s-m_Z^2)^2 + (m_Z\Gamma_Z)^2} \\ &\quad \times \left\{(Q_{zR,e}^2 Q_{zR,q}^2 + Q_{zL,e}^2 Q_{zL,q}^2) + 7(Q_{zR,e}^2 Q_{zL,q}^2 + Q_{zL,e}^2 Q_{zR,q}^2)\right\},\end{aligned} \tag{29.35}$$

$$\sigma_\gamma(e^+e^- \to \bar{q}q) = \frac{Q_q^2 e^4}{4\pi s}, \tag{29.36}$$

and [2]

$$\begin{aligned}
\sigma_{\text{interfere}}(e^+e^- \to \bar{q}q) &= -\frac{1}{64\pi}\frac{Q_q e^4}{\sin^2\theta_w\cos^2\theta_w}\left|\frac{1}{s-m_Z^2+im_Z\Gamma_Z}+\frac{1}{s-m_Z^2-im_Z\Gamma_Z}\right| \\
&\quad\times\left\{4(Q_{zR,e}+Q_{zL,e})(Q_{zR,q}+Q_{zL,q})-3(Q_{zR,e}-Q_{zL,e})(Q_{zR,q}-Q_{zL,q})\right\} \\
&= -\frac{1}{32\pi}\frac{Q_q e^4}{\sin^2\theta_w\cos^2\theta_w}\frac{s-m_Z^2}{(s-m_Z^2)^2+(m_Z\Gamma_Z)^2} \\
&\quad\times\left\{(Q_{zR,e}Q_{zR,q}+Q_{zL,e}Q_{zL,q})+7(Q_{zR,e}Q_{zL,q}+Q_{zL,e}Q_{zR,q})\right\}
\end{aligned} \tag{29.37}$$

Then, the total cross section is

$$\sigma_{\text{tot}}(e^+e^- \to \bar{q}q) = \sum_q \left(\sigma_Z + \sigma_\gamma + \sigma_{\text{interfere}}\right). \tag{29.38}$$

The sum is over the quark flavors listed in Table 29.1. Since the Z-boson and the photon does not mix off-diagonal generations, this sum if flavor diagonal.

(b) **This question asks about 1-loop, not just NLO. Need more works. As for now, just focus on $\boldsymbol{\sigma_R}$ and $\boldsymbol{\sigma_V}$**

At NLO, the diagram involves the real emission of a gluon in the final state and a vertex correction involving the outgoing quarks and a virtual gluon. One can in principle follows the procedures of Chapter 20.A of textbook, and notice that the correction on a Z diagram can be seen essentially as a photon diagram, but just with proper replacement of $e \to \frac{e}{\sin\theta_w\cos\theta_w}$, electric charge $Q \to Q_L, Q_R$ of which left-handed and right-handed particles just couple with the Z by a different charge, and the propagator $-i\frac{g^{\mu\nu}}{p^2} \to -i\frac{g^{\mu\nu}}{p^2-m_Z^2}$. However, the QCD corrections are the same for left- or right-handed quarks since QCD is non-chiral and the calculations for the FSR and vertex correction diagrams have nothing to do with the propagator replacement. Therefore, one just expects the same results of what's already calculated in Chapter 26.3 of textbook with proper replacements of coupling strength and charges mentioned above. The same is also true for the interference terms because the corrections apply in the same way for the Z diagram and the photon diagram.

Therefore, at NLO, we simply expects

$$\sigma_{\text{NLO}}(e^+e^- \to \bar{q}q) = \sigma_0\left(1+\frac{3\alpha_s}{4\pi}C_{\text{F}}\right) = \sigma_0\left(1+\frac{\alpha_s}{\pi}\right), \tag{29.39}$$

where $\sigma_0(e^+e^- \to \bar{q}q)$ is given by Eq. (29.38).

(c) Using the values $m_Z = 91.7816$ GeV, $\Gamma_Z = 2.4952$ GeV, $e = 0.303$, and $\alpha_s/\pi = 0.035$, $\sin^2\theta_w = 0.223$, as well as charges given in Table 29.1, we plotted the cross sections at NLO in Fig. 29.1. We also plotted the effect of interference term in Fig. 29.2. From the figures, the interference can either be ignored when either a) $s \ll m_Z$ where photon diagram dominates or b) $s \sim m_Z$ where Z diagram dominates.

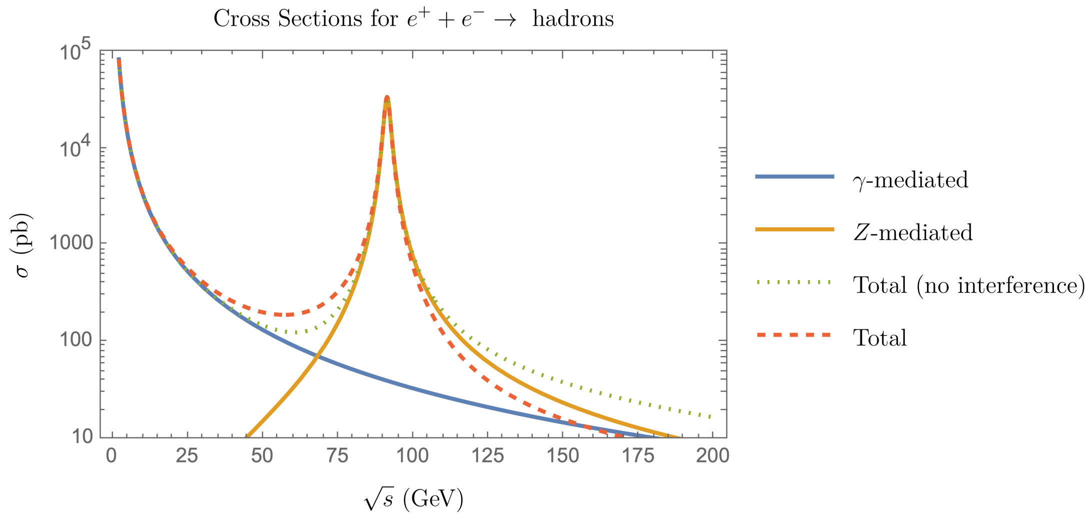


Fig. 29.1: Cross sections as a function of center-of-mass energy $\sqrt{s}$.

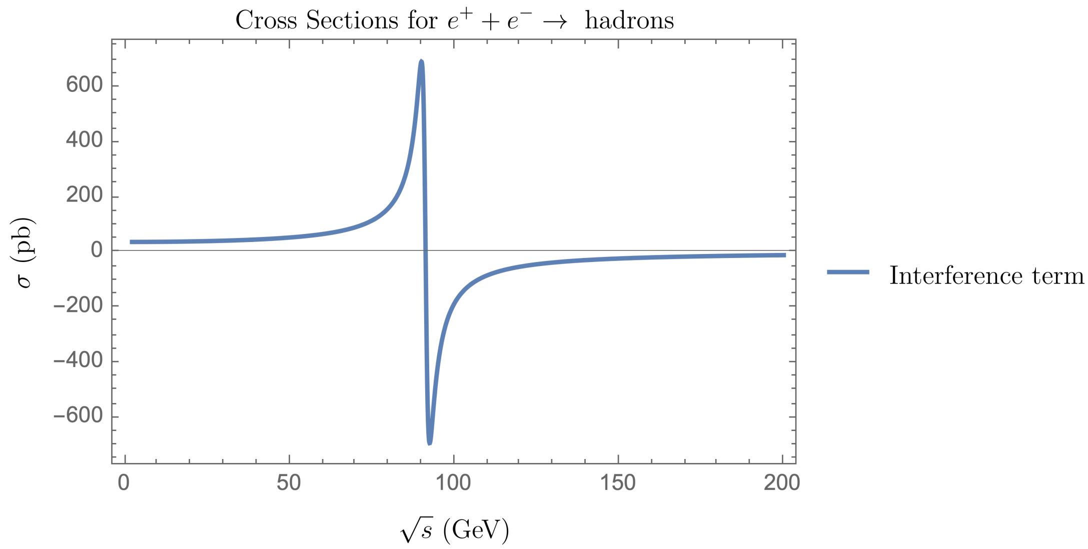


Fig. 29.2: The amount of interference as a function of center-of-mass energy $\sqrt{s}$.

29.3

(a)

(b)

(c)

(d)

[2]Technically, $\sigma_{\text{interfere}}(e^+e^- \to \bar{q}q)$ is of course not a cross section since it could run to negative value. See Fig. 29.2. Forgive me to use sloppy notations here.

29.4

(a)

(b)

(c)

Chapter 30

Anomalies

30.1

$$\begin{aligned} iM_5^{\alpha\mu\nu\rho} &= -i^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[\gamma^\mu P_L \frac{\not{k}}{k^2}\gamma^\nu P_L \frac{\not{k}+\not{q}_2}{(k+q_2)^2}\gamma^\rho P_L \frac{\not{k}+\not{q}_2+\not{q}_3}{(k+q_2+q_3)^2}\gamma^\alpha\gamma^5 \frac{\not{k}-\not{q}_1}{(k-q_1)^2} + (6 \text{ perms})\right] \\ &= \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[\gamma^\mu \frac{\not{k}}{k^2}\gamma^\nu \frac{\not{k}+\not{q}_2}{(k+q_2)^2}\gamma^\rho \frac{\not{k}+\not{q}_2+\not{q}_3}{(k+q_2+q_3)^2}\gamma^\alpha\gamma^5 \frac{\not{k}-\not{q}_1}{(k-q_1)^2} + (6 \text{ perms})\right]. \end{aligned} \tag{30.1}$$

Using $p^\mu = q_1^\mu + q_2^\mu$ so that

$$\not{p}\gamma^5 = (\not{q}_1 + \not{q}_2 + \not{q}_3) = \gamma^5(\not{k} - \not{q}_1) + (\not{k} + \not{q}_2 + \not{q}_3)\gamma^5. \tag{30.2}$$

Then,

$$\begin{aligned} p_\alpha M_5^{\alpha\mu\nu\rho} &= \int \frac{d^4k}{(2\pi)^4} \left[\frac{\operatorname{Tr}\left[\gamma^\mu \not{k}\gamma^\nu(\not{k}+\not{q}_2)\gamma^\rho(\not{k}+\not{q}_2+\not{q}_3)\gamma^5\right]}{k^2(k+q_2)^2(k+q_2+q_3)^2} \right. \\ &\qquad \left. + \frac{\operatorname{Tr}\left[\gamma^\mu \not{k}\gamma^\nu(\not{k}+\not{q}_2)\gamma^\rho(\not{k}-\not{q}_1)\gamma^5\right]}{k^2(k+q_2)^2(k-q_1)^2} + (6 \text{ perms})\right] \\ &= -2\int \frac{d^4k}{(2\pi)^4} \left[\frac{(k+q_2)^\nu \operatorname{Tr}\left[\gamma^\mu \not{k}(\not{q}_2+\not{q}_3)\gamma^\rho\gamma^5\right] + (k+q_2+q_3)^\rho \operatorname{Tr}\left[\gamma^\mu \not{k}\not{q}_2\gamma^\nu\gamma^5\right]}{k^2(k+q_2)^2(k+q_2+q_3)^2} \right. \\ &\qquad \left. - \frac{(k+q_2)^\nu \operatorname{Tr}\left[\gamma^\mu \not{k}\not{q}_1\gamma^\rho\gamma^5\right] - (k-q_1)^\rho \operatorname{Tr}\left[\gamma^\mu \not{k}\not{q}_2\gamma^\nu\gamma^5\right]}{k^2(k+q_2)^2(k-q_1)^2} + (6 \text{ perms})\right] \end{aligned} \tag{30.3}$$

30.2

From Eq. (29.63) of the textbook,

$$\mathcal{L} = -Y_{ij}^e \bar{L}^i H e_R^j - Y_{ij}^\nu \bar{L}^i \tilde{H} \nu_R^j - iM_{ij}(\nu_R^i)^c \nu_R^j + h.c., \tag{30.4}$$

one can assign $L_{\nu_R} = 1$ and $L_L = 1$ ($L_{\bar{L}} = -1$) such that the Dirac neutrino masses are anomalous-free at the Lagrangian level, but Majorana neutrino masses violates L by 2 units. However, the lepton number L is always anomalous in SM by the chiral anomaly following the discussion in Chapter 30.5 of the textbook. **Thus, L is anomalous for both Dirac and Majorana neutrino masses.**

On the other hand, using the same logic, for $B - L$, it is again anomalous at the Lagrangian level for the Majorana neutrino masses. However, it is not a global anomaly following the discussion above Eq. (30.87) of the textbook. **Thus, $B-L$ is only anomalous for Majorana neutrino masses. In fact, any $U(1)$ charges are anomalous for Majorana neutrino masses.**

30.3

Chapter 31

Precision tests of the Standard Model

31.1

The relevant Lagrangian of the 4-Fermi theory is given by Eq. (23.40) and Eq. (29.72) of the textbook:

$$\mathcal{L}_{4F} = -\frac{4G_F}{\sqrt{2}}\bar{\psi}_{\nu_\mu}\gamma^\mu P_L \psi_\mu \bar{\psi}_e \gamma^\mu P_L \psi_{\nu_e} + h.c.. \tag{31.1}$$

We shall use p_1 to denote the 4-momentum of the incoming μ^-, p_2, p_3, p_4 to denote the 4-momentum of the outgoing $\bar{\nu}_e$, ν_μ and e^-, respectively. Then, the amplitude is given by

$$i\mathcal{M} = -i\frac{4G_F}{\sqrt{2}}\left[\bar{u}_3\gamma^\mu P_L u_1\right]\left[\bar{u}_4\gamma^\mu P_L v_2\right], \tag{31.2}$$

where we used shorthands $u_i = u(p_i)$ and $v_i = v(p_i)$. Treating the neutrinos as massless but still keep the electron as massive, we can get the amplitude square after spin sum as

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}|^2 &= 8G_F^2 \,\mathrm{Tr}\Big[(\not{p}_1 + m_\mu)\gamma^\mu P_L \not{p}_3 \gamma^\nu P_L\Big]\,\mathrm{Tr}\Big[(\not{p}_4 + m_e)\gamma^\mu P_L \not{p}_2 \gamma^\nu P_L\Big] \\
&= 8G_F^2 \,\mathrm{Tr}\Big[\not{p}_1 \gamma^\mu \not{p}_3 \gamma^\nu P_L\Big]\,\mathrm{Tr}\Big[\not{p}_4 \gamma^\mu \not{p}_2 \gamma^\nu P_L\Big] \\
&= 32G_F^2 \Big[(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}p_{13}) - i\varepsilon^{\mu\nu\alpha\beta}p_1^\alpha p_3^\beta\Big]\Big[(p_4^\mu p_2^\nu + p_4^\nu p_2^\mu - g^{\mu\nu}p_{24}) + i\varepsilon^{\mu\nu\delta\sigma}p_2^\delta p_4^\sigma\Big] \\
&= 32G_F^2\left[2p_{14}p_{23} + 2p_{12}p_{34} + 2(p_{12}p_{34} - p_{14}p_{23})\right] \\
&= 64G_F^2 p_{12}p_{34},
\end{aligned} \tag{31.3}$$

where we used the shorthand $p_{ij} \equiv p_i \cdot p_j$, and also the facts $\left\{\gamma^\mu, \gamma^5\right\} = 0$ and the anti-symmetric property of the Levi-Civita tensor.

Next, if we focus on the CM frame of the decayed muon, we shall get $p_1 = (m_\mu, 0)$, $p_2 = (E, \vec{p}_2)$, of which $E = |\vec{p}_2|$, and $p_1 = p_2 + p_3 + p_4$. Then,

$$p_{12} = m_\mu E, \tag{31.4}$$

$$(p_1 - p_2)^2 = (p_3 + p_4)^2 \implies p_{34} = \frac{m_\mu^2 - m_e^2 - 2m_\mu E}{2}, \tag{31.5}$$

such that

$$|\mathcal{M}|^2 = 32G_F^2(m_\mu^2 - m_e^2 - 2m_\mu E)m_\mu E. \tag{31.6}$$

Notice that if we ignore the electron mass, we get the amplitude square given in problem 5.3 of the textbook.

We can use the 3-body decay phase space integral Eq. (20.42) of the textbook (derived in Problem (20.1)):

$$\int d\Pi_{\text{LIPS}} = \frac{Q^2}{128\pi^3}\int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\frac{\beta}{1-x_1}} dx_2 \tag{31.7}$$

with proper replacement $Q \to m_\mu$, $x_1 = 2\frac{E}{m_\mu}$, $x_2 = 2\frac{E_{\nu_\mu}}{m_\mu}$, $\beta \to r$. The amplitude square, expressed with these dimensionless variables, becomes

$$|\mathcal{M}|^2 = 16G_F^2 m_\mu^4(1 - r - x_1)x_1. \tag{31.8}$$

Using the decay rate formula of Eq. (5.24) from the textbook and focusing on the rest frame of the decayed muon,

$$\begin{aligned}
\Gamma &= \frac{1}{2m_\mu}\int |\mathcal{M}|^2 d\Pi_{\text{LIPS}} \\
&= G_F^2\frac{m_\mu^5}{16\pi^3}\int_0^{1-r} dx_1[(1-r-x_1)x_1]\int_{1-x_1-r}^{1-\frac{r}{1-x_1}} dx_2 \\
&= G_F^2\frac{m_\mu^5}{16\pi^3}\int_0^{1-r} dx_1\frac{x_1^2(1-r-x_1)^2}{1-x_1} \\
&= G_F^2\frac{m_\mu^5}{192\pi^3}\left(1 - 8r + 8r^3 - r^4 - 12r^2\ln r\right).
\end{aligned} \tag{31.9}$$

31.2

(a)

(b)

(c)

(d)

Chapter 32

Quantum chromodynamics and the parton model

32.1

By definition,

$$F_1(0) = -Q = -\int d^3x\rho(x). \tag{32.1}$$

Notice

$$\begin{aligned}
F(q^2) &= \int d^3x e^{i\vec{q}\cdot\vec{x}} V(x) \\
&= 2\pi \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) e^{iqr\cos\theta} V(x) \\
&= 2\pi \int_0^\infty r^2 dr \frac{1}{iqr} \left(e^{iqr} - e^{-iqr}\right) V(x) \\
&= 4\pi \int_0^\infty r^2 dr \frac{1}{qr} \sin(qr) V(x) \\
&= 4\pi \int_0^\infty r^2 dr \frac{1}{qr} \left[qr - \frac{1}{6}(qr)^3 + \cdots\right] V(x) \\
&= 4\pi \int_0^\infty r^2 dr \left[1 - \frac{1}{6}(qr)^2 + \cdots\right] V(x),
\end{aligned} \tag{32.2}$$

Now, we can observe that,

$$\begin{aligned}
\left.\frac{dF(q^2)}{dq^2}\right|_{q^2=0} &= \frac{4\pi}{6} \int_0^\infty r^4 \rho(x) dr \\
&= \frac{1}{6} \int d^3x r^2 \rho(x) \\
&= \frac{1}{6} \langle r^2 \rangle.
\end{aligned} \tag{32.3}$$

From Eq. (32.9) of the textbook,

$$F_1(q^2) \sim \frac{1}{\left(1 - \frac{q^2}{0.71\ \mathrm{GeV}^2}\right)^2}, \tag{32.4}$$

the proton's mean charge radius is

$$\langle r_p^2\rangle = 6\frac{dF_1(q^2)}{dq^2}\bigg|_{q^2=0} = \frac{12}{0.71\ \text{GeV}^2} \approx 16.9\ \text{GeV}^{-2} = 16.9\times 3.894\times 10^{-32}\ \text{m}^2 = 0.658\ \text{fm}^2, \tag{32.5}$$

or taking the square root,

$$r_p^{rms} = \sqrt{\langle r_p^2\rangle} \approx 0.81\ \text{fm}. \tag{32.6}$$

32.2

The parton momentum can be viewed as the sum of the average momentum of each constituent. Also, since the PDFs are interpreted as classical probabilities, we have

$$P^\mu = \sum_j \langle p_j^\mu\rangle = \sum_j \int_0^1 d\xi p_j^\mu f_j(\xi) = \sum_j \int_0^1 d\xi P^\mu \xi_j f_j(\xi), \tag{32.7}$$

where we used $p_j^\mu = \xi P^\mu$. Eliminating P^μ from both sides, we arrived at

$$\sum_j \int_0^1 d\xi[\xi_j f_j(\xi)] = 1. \tag{32.8}$$

32.3

As a distribution, the Eq. (32.38) can be defined through

$$\int_0^1 dz\frac{f(z)}{(1-z)^{1+\varepsilon}} = -\frac{f(1)}{\varepsilon} + \int_0^1 dz\frac{f(z)-f(1)}{1-z} + \sum_{n=1}^{\infty}\frac{(-\varepsilon)^n}{n!}\int_0^1 dz(f(z)-f(1))\frac{\ln^n(1-z)}{1-z}. \tag{32.9}$$

Let $x = 1-z$, and notice that ε above is used to regulate the IR divergence, and thus $\varepsilon < 0$, the Eq. (32.118) of the textbook can thus be written as

$$\int_0^1 dz(1-z)^{-1-\varepsilon}f(z) = \int_0^1 dz(1-z)^{-1-\varepsilon}f(1) + \int_0^1 dz(1-z)^{-1-\varepsilon}[f(z)-f(1)]. \tag{32.10}$$

The first term is evaluated to be

$$\int_0^1 dz(1-z)^{-1-\varepsilon}f(1) = \left[\frac{1}{\varepsilon}(1-z)^{-\varepsilon}f(1)\right]_0^1 = -\frac{f(1)}{\varepsilon}, \tag{32.11}$$

where we used the fact that $\varepsilon < 0$. The second term is

$$\begin{aligned}\int_0^1 dz(1-z)^{-1-\varepsilon}[f(z)-f(1)] &= \int_0^1 dz\frac{e^{-\varepsilon\ln(1-z)}}{1-z}[f(z)-f(1)]\\ &= \int_0^1 dz\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{(-\varepsilon)^n\ln^n(1-z)}{n!}[f(z)-f(1)]\\ &= \int_0^1 dz\frac{f(z)-f(1)}{1-z} + \sum_{n=1}^{\infty}\frac{(-\varepsilon)^n}{n!}\int_0^1 dz(f(z)-f(1))\frac{\ln^n(1-z)}{1-z}.\end{aligned} \tag{32.12}$$

Thus, we derived the expansion of Eq. (32.38) of the textbook:

$$\frac{1}{(1-z)^{1+\varepsilon}} = -\frac{1}{\varepsilon}\delta(1-z) + \frac{1}{[1-z]_+} - \varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + \sum_{n=2}^{\infty}\frac{(-\varepsilon)^n}{n!}\left[\frac{\ln^n(1-z)}{1-z}\right]_+ . \quad (32.13)$$

32.4

32.5

32.6

Part V

Advanced topics

Chapter 33

Effective actions and Schwinger proper time

33.1

The general Lorentz-invariant expression for the effective Lagrangian for any constant $F_{\mu\nu}$ can be written as

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s}e^{-s\varepsilon}e^{-ism^2}\frac{\operatorname{Re}\cos(esX)}{\operatorname{Im}\cos(esX)}F_{\mu\nu}\tilde{F}_{\mu\nu}, \tag{33.1}$$

where X is a scalar function of the electric and magnetic fields defined by

$$X \equiv \sqrt{\frac{1}{2}F_{\mu\nu}^2 + \frac{i}{2}F_{\mu\nu}\tilde{F}_{\mu\nu}}, \tag{33.2}$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Chapter 34

Background fields

34.1

Chapter 35

Heavy-quark physics

35.1

Chapter 36

Jets and effective field theory

36.1

Appendices ✓

Appendix A

Conventions ✓

A.1

(a) Since the action S must be dimensionless, the Lagrangian density must carry mass dimension $[\mathcal{L}] = d$. Given that $[\partial_\mu] = 1$ (cf. Eq. (A.4) of the textbook), we proceed to determine the dimensions of the fields and couplings.

From the term $-\frac{1}{4}F_{\mu\nu}^2$, we have

$$\begin{aligned} 2 \times [F_{\mu\nu}] &= [\mathcal{L}] \\ 2 \times (1 + [A_\mu]) &= d \\ [A_\mu] &= \frac{d}{2} - 1. \end{aligned} \tag{A.1}$$

From the term $-\phi^* \Box \phi$, we have

$$\begin{aligned} 2 \times [\phi] + 2 &= [\mathcal{L}] \\ 2 \times [\phi] + 2 &= d \\ [\phi] &= \frac{d}{2} - 1. \end{aligned} \tag{A.2}$$

From the term $gA_\mu \phi^* \partial_\mu \phi$, we have

$$\begin{aligned} [g] + [A_\mu] + 2 \times [\phi] + 1 &= [\mathcal{L}] \\ [g] + 3 \times \left(\frac{d}{2} - 1\right) + 1 &= d \\ [g] &= 2 - \frac{d}{2}. \end{aligned} \tag{A.3}$$

Finally, from the term $\lambda \phi^3$, we have

$$\begin{aligned} [\lambda] + 3 \times [\phi] &= [\mathcal{L}] \\ [\lambda] + 3 \times \left(\frac{d}{2} - 1\right) &= d \\ [\lambda] &= 3 - \frac{d}{2}. \end{aligned} \tag{A.4}$$

(b) To ensure that the electromagnetic coupling g is renormalizable, we require from Eq. (A.3):

$$0 = [g] = 2 - \frac{d}{2} \implies d = 4. \tag{A.5}$$

Similarly, for the ϕ^3 scalar self-interaction coupling λ to be renormalizable, Eq. (A.4) yields:

$$0 = [\lambda] = 3 - \frac{d}{2} \implies d = 6. \tag{A.6}$$

Appendix B

Regularization ✓

B.1

Starting with

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n}, \tag{B.1}$$

where $\Delta < 0$. Now the integral still have poles at $k_0 = \sqrt{\vec{k}^2 + \Delta} - i\epsilon$ and $k_0 = -\sqrt{\vec{k}^2 + \Delta} + i\epsilon$. If $|\vec{k}^2| > |\Delta|$, the poles will then just on the same quadrants of the k_0 complex plane as the case if $\Delta > 0$. Thus, let's assume $|\vec{k}^2| < |\Delta|$.

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k_0^2 + (-\vec{k}^2 - \Delta + i\epsilon))^n}. \tag{B.2}$$

The poles are now at $k_0 = i(\sqrt{-\vec{k}^2 - \Delta} + i\epsilon) = -\epsilon + i\sqrt{-\vec{k}^2 - \Delta}$ and $k_0 = i(-\sqrt{-\vec{k}^2 - \Delta} - i\epsilon) = \epsilon - i\sqrt{-\vec{k}^2 - \Delta}$. Since $\epsilon > 0$ and $\sqrt{-\vec{k}^2 - \Delta} > 0$, the poles are still in the top-left and bottom-right quadrants of the k_0 complex plane, for which the integral over the figure-eight contour still vanishes and the conclusion that the integral over the real and the imaginary axis are equal and opposite still holds.

Bibliography

[1] M. D. Schwartz, Quantum Field Theory and the Standard Model, Cambridge University Press, 2014.

[2] R. S. Park, W. M. Folkner, A. S. Konopliv, J. G. Williams, D. E. Smith, M. T. Zuber, Precession of Mercury's Perihelion from Ranging to the MESSENGER Spacecraft, Astron. J. 153 (3) (2017) 121. doi:10.3847/1538-3881/aa5be2.

[3] B. P. Abbott, et al., Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116 (6) (2016) 061102. arXiv:1602.03837, doi:10.1103/PhysRevLett.116.061102.

[4] S. Weinberg, Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass, Phys. Rev. 135 (1964) B1049–B1056. doi:10.1103/PhysRev.135.B1049.

[5] S. Ferrara, E. Remiddi, Absence of the Anomalous Magnetic Moment in a Supersymmetric Abelian Gauge Theory, Phys. Lett. B 53 (1974) 347–350. doi:10.1016/0370-2693(74)90399-2.

[6] D. P. Aguillard, et al., Measurement of the Positive Muon Anomalous Magnetic Moment to 0.20 ppm, Phys. Rev. Lett. 131 (16) (2023) 161802. arXiv:2308.06230, doi:10.1103/PhysRevLett.131.161802.

[7] X. Fan, T. G. Myers, B. A. D. Sukra, G. Gabrielse, Measurement of the Electron Magnetic Moment, Phys. Rev. Lett. 130 (7) (2023) 071801. arXiv:2209.13084, doi:10.1103/PhysRevLett.130.071801.

[8] M. Campostrini, M. Hasenbusch, A. Pelissetto, E. Vicari, The Critical exponents of the superfluid transition in He-4, Phys. Rev. B 74 (2006) 144506. arXiv:cond-mat/0605083, doi:10.1103/PhysRevB.74.144506.

[9] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, H. S. Shao, T. Stelzer, P. Torrielli, M. Zaro, The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations, JHEP 07 (2014) 079. arXiv:1405.0301, doi:10.1007/JHEP07(2014)079.